LAPLACE TRANSFORMS

Introduction to Laplace transforms

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Introduction

Why it is important to understand: **Introduction to Laplace transforms**

- **The Laplace transform is a very powerful mathematical tool applied in various areas of engineering and science. With the increasing complexity of engineering problems, Laplace transforms help in solving complex problems with a very simple approach; the transform is an integral transform method which is particularly useful in solving linear ordinary differential equations. It has very wide applications in various areas of physics, electrical engineering, control engineering, optics, mathematics and signal processing. This chapter just gets us started in understanding some standard Laplace transforms.**

At the end of this chapter, you should be able to:

- define a Laplace transform
- recognise common notations used for the Laplace transform
- derive Laplace transforms of elementary functions
- use a standard list of Laplace transforms to determine the transform of common functions

67.1 Introduction

The solution of most electrical circuit problems can be reduced ultimately to the solution of differential equations. The use of **Laplace**∗ **transforms** provides an alternative method for solving linear differential equations.

67.2 Definition of a Laplace transform

The Laplace transform of the function $f(t)$ is defined by the integral $\int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t$, where s is a parameter assumed to be a real number.

Common notations used for the Laplace transform

There are various commonly used notations for the Laplace transform of $f(t)$ and these include:

(i) $\mathcal{L}\{f(t)\}$ or $L\{f(t)\}$

(ii) $\mathcal{L}(f)$ or Lf

(iii) $\overline{f}(s)$ or $f(s)$

Also, the letter p is sometimes used instead of s as the parameter. The notation adopted in this book will be $f(t)$ for the original function and $\mathcal{L}\{f(t)\}$ for its Laplace transform.

Hence, from above:

$$\mathcal{L}\{f(t)\} = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t \qquad (1)$$

67.3 Linearity property

From equation (1),

$$\mathcal{L}\{k f(t)\} = \int_0^\infty e^{-st} k\, f(t)\, dt$$
$$= k \int_0^\infty e^{-st} f(t)\, dt$$

i.e. $\mathcal{L}\{k\, f(t)\} = k\mathcal{L}\{f(t)\}$ (2)

where k is any constant.

Similarly,

$$\mathcal{L}\{a\, f(t) + bg(t)\} = \int_0^\infty e^{-st}(a\, f(t) + bg(t))\, dt$$
$$= a \int_0^\infty e^{-st} f(t)\, dt$$
$$+ b \int_0^\infty e^{-st} g(t)\, dt$$

i.e. $\mathcal{L}\{a f(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$, (3)

where a and b are any real constants.

The Laplace transform is termed a **linear operator**

67.3 Linearity property

From equation (1),

$$\mathcal{L}\{kf(t)\} = \int_0^\infty e^{-st} k\, f(t)\, dt$$
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where k is any constant.

Similarly,

$$\mathcal{L}\{a\, f(t) + bg(t)\} = \int_0^\infty e^{-st}(a\, f(t) + bg(t))\, dt$$
$$= a\int_0^\infty e^{-st} f(t)\, dt + b\int_0^\infty e^{-st} g(t)\, dt$$

i.e. $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\},$ (3)

where a and b are any real constants.
The Laplace transform is termed a **linear operator**

67.4 Laplace transforms of elementary functions

(a) $\boldsymbol{f(t)=1}$. From equation (1),

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= -\frac{1}{s}[e^{-s(\infty)} - e^0] = -\frac{1}{s}[0-1]$$

$$= \frac{\mathbf{1}}{\boldsymbol{s}} \text{ (provided } s > 0)$$

(b) $\boldsymbol{f(t)=k}$. From equation (2),

$$\mathcal{L}\{k\} = k\mathcal{L}\{1\}$$

Hence $\mathcal{L}\{k\} = k\left(\frac{1}{s}\right) = \frac{\boldsymbol{k}}{\boldsymbol{s}}$, from (a) above.

67.4 Laplace transforms of elementary functions

(c) $\boldsymbol{f(t) = e^{at}}$ (where a is a real constant $\neq 0$).
From equation (1),

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}(e^{at})\,dt = \int_0^\infty e^{-(s-a)t}\,dt,$$

from the laws of indices,

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{-(s-a)}(0-1)$$

$$= \boldsymbol{\frac{1}{s-a}}$$

(provided $(s-a) > 0$, i.e. $s > a$)

(d) $\boldsymbol{f(t) = \cos at}$ (where a is a real constant).
From equation (1),

$$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st}\cos at\,dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(a\sin at - s\cos at)\right]_0^\infty$$

67.4 Laplace transforms of elementary functions

(d) $\boldsymbol{f(t) = \cos at}$ (where a is a real constant).

From equation (1),

$$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st} \cos at \, dt$$

$$= \left[\frac{e^{-st}}{s^2 + a^2} (a \sin at - s \cos at) \right]_0^\infty$$

by integration by parts twice (see page 485),

$$= \left[\frac{e^{-s(\infty)}}{s^2 + a^2} (a \sin a(\infty) - s \cos a(\infty)) - \frac{e^0}{s^2 + a^2} (a \sin 0 - s \cos 0) \right]$$

$$= \frac{s}{s^2 + a^2} \text{ (provided } s > 0)$$

(e) $\boldsymbol{f(t) = t}$. From equation (1),

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} t \, dt = \left[\frac{te^{-st}}{-s} - \int \frac{e^{-st}}{-s} dt \right]_0^\infty$$

67.4 Laplace transforms of elementary functions

$$= \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^{\infty}$$

by integration by parts

tích phân từng phần

$$= \left[\frac{\infty e^{-s(\infty)}}{-s} - \frac{e^{-s(\infty)}}{s^2} \right] - \left[0 - \frac{e^0}{s^2} \right]$$

$$= (0-0) - \left(0 - \frac{1}{s^2} \right)$$

since $(\infty \times 0) = 0$

$$= \frac{\mathbf{1}}{\mathbf{s^2}} \text{ (provided } s > 0)$$

(f) $\boldsymbol{f(t) = t^n}$ (where $n = 0, 1, 2, 3, \ldots$).

By a similar method to (e) it may be shown that $\mathcal{L}\{t^2\} = \frac{2}{s^3}$ and $\mathcal{L}\{t^3\} = \frac{(3)(2)}{s^4} = \frac{3!}{s^4}$. These results can be extended to n being any positive integer.

Thus $\boldsymbol{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}}$ provided $s > 0$)

67.4 Laplace transforms of elementary functions

(g) $\boldsymbol{f(t) = \sinh at}$. From Chapter 16, $\sinh at = \frac{1}{2}(e^{at} - e^{-at})$. Hence,

$$\mathcal{L}\{\sinh at\} = \mathcal{L}\left\{\frac{1}{2}e^{at} - \frac{1}{2}e^{-at}\right\}$$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\} - \frac{1}{2}\mathcal{L}\{e^{-at}\}$$

from equations (2) and (3)

$$= \frac{1}{2}\left[\frac{1}{s-a}\right] - \frac{1}{2}\left[\frac{1}{s+a}\right]$$

from (c) above

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

67.5 Worked problems on standard Laplace transforms

	Function $f(t)$	Laplace transforms $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$
(i)	1	$\frac{1}{s}$
(ii)	k	$\frac{k}{s}$
(iii)	e^{at}	$\frac{1}{s-a}$
(iv)	$\sin at$	$\frac{a}{s^2+a^2}$
(v)	$\cos at$	$\frac{s}{s^2+a^2}$
(vi)	t	$\frac{1}{s^2}$
(vii)	t^2	$\frac{2!}{s^3}$
(viii)	$t^n\ (n = 1, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$
(ix)	$\cosh at$	$\frac{s}{s^2-a^2}$
(x)	$\sinh at$	$\frac{a}{s^2-a^2}$

Problem 1. Using a standard list of Laplace transforms, determine the following:

(a) $\mathcal{L}\left\{1 + 2t - \frac{1}{3}t^4\right\}$ (b) $\mathcal{L}\{5e^{2t} - 3e^{-t}\}$

(a) $\mathcal{L}\left\{1 + 2t - \frac{1}{3}t^4\right\}$

$$= \mathcal{L}\{1\} + 2\mathcal{L}\{t\} - \frac{1}{3}\mathcal{L}\{t^4\}$$

from equations (2) and (3)

$$= \frac{1}{s} + 2\left(\frac{1}{s^2}\right) - \frac{1}{3}\left(\frac{4!}{s^{4+1}}\right)$$

from (i), (vi) and (viii) of Table 67.1

$$= \frac{1}{s} + \frac{2}{s^2} - \frac{1}{3}\left(\frac{4.3.2.1}{s^5}\right)$$

$$= \frac{1}{s} + \frac{2}{s^2} - \frac{8}{s^5}$$

(b) $\mathcal{L}\{5e^{2t} - 3e^{-t}\} = 5\mathcal{L}(e^{2t}) - 3\mathcal{L}\{e^{-t}\}$

from equations (2) and (3)

$$= 5\left(\frac{1}{s-2}\right) - 3\left(\frac{1}{s-(-1)}\right)$$

67.5 Worked problems on standard Laplace transforms

Problem 3. Prove that

(a) $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$ (b) $\mathcal{L}\{t^2\} = \dfrac{2}{s^3}$

(c) $\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2-a^2}$

(a) From equation (1),

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty$$

by integration by parts

$$= \frac{1}{s^2+a^2}[e^{-s(\infty)}(-s\sin a(\infty) - a\cos a(\infty)) - e^0(-s\sin 0 - a\cos 0)]$$

$$= \frac{1}{s^2+a^2}[(0) - 1(0-a)]$$

$$= \frac{a}{s^2+a^2} \text{ (provided } s > 0)$$

(b) From equation (1),

$$\mathcal{L}\{t^2\} = \int_0^\infty e^{-st} t^2 \, dt$$

$$= \left[\frac{t^2 e^{-st}}{-s} - \frac{2te^{-st}}{s^2} - \frac{2e^{-st}}{s^3}\right]_0^\infty$$

by integration by parts twice

$$= \left[(0-0-0) - \left(0 - 0 - \frac{2}{s^3}\right)\right]$$

$$= \frac{2}{s^3} \text{ (provided } s > 0)$$

67.5 Worked problems on standard Laplace transforms

Problem 3. Prove that

(a) $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$ (b) $\mathcal{L}\{t^2\} = \dfrac{2}{s^3}$

(c) $\mathcal{L}\{\cosh at\} = \dfrac{s}{s^2-a^2}$

(a) From equation (1),

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st}\sin at\,dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty$$

by integration by parts

$$= \frac{1}{s^2+a^2}[e^{-s(\infty)}(-s\sin a(\infty) - a\cos a(\infty)) - e^0(-s\sin 0 - a\cos 0)]$$

(c) From equation (1),

$$\mathcal{L}\{\cosh at\} = \mathcal{L}\left\{\frac{1}{2}(e^{at}+e^{-at})\right\}$$

from Chapter 16

$$= \frac{1}{2}\mathcal{L}\{e^{at}\} + \frac{1}{2}\mathcal{L}\{e^{-at}\}$$ from equations (2) and (3)

$$= \frac{1}{2}\left(\frac{1}{s-a}\right) + \frac{1}{2}\left(\frac{1}{s-(-a)}\right)$$

from (iii) of Table 67.1

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{(s+a)+(s-a)}{(s-a)(s+a)}\right]$$

$$= \frac{s}{s^2-a^2}$$ (provided $s > a$)

67.5 Worked problems on standard Laplace transforms

1. (a) $2t-3$ (b) $5t^2+4t-3$

2. (a) $\frac{t^3}{24}-3t+2$ (b) $\frac{t^5}{15}-2t^4+\frac{t^2}{2}$

3. (a) $5e^{3t}$ (b) $2e^{-2t}$

4. (a) $4\sin 3t$ (b) $3\cos 2t$

5. (a) $7\cosh 2x$ (b) $\frac{1}{3}\sinh 3t$

6. (a) $2\cos^2 t$ (b) $3\sin^2 2x$

7. (a) $\cosh^2 t$ (b) $2\sinh^2 2\theta$

8. $4\sin(at+b)$, where a and b are constants.

9. $3\cos(\omega t-\alpha)$, where ω and α are constants.

10. Show that $\mathcal{L}(\cos^2 3t-\sin^2 3t)=\frac{s}{s^2+36}$

LAPLACE TRANSFORMS

Properties of Laplace transforms

Introduction

Why it is important to understand: **Properties of Laplace transforms**

- **As stated in the preceding chapter, the Laplace transform is a widely used integral transform with many applications in engineering, where it is used for analysis of linear time-invariant systems such as electrical circuits, harmonic oscillators, optical devices, and mechanical systems. The Laplace transform is also a valuable tool in solving differential equations, such as in electronic circuits, and in feedback control systems, such as in stability and control of aircraft systems.**
- **This chapter considers further transforms together with the Laplace transform of derivatives that are needed when solving differential equations.**

At the end of this chapter, you should be able to:

- derive the Laplace transform of $e^{at}f(t)$
- use a standard list of Laplace transforms to determine transforms of the form $e^{at}f(t)$
- derive the Laplace transforms of derivatives
- state and use the initial and final value theorems

67.1 Introduction

The solution of most electrical circuit problems can be reduced ultimately to the solution of differential equations. The use of **Laplace**∗ **transforms** provides an alternative method for solving linear differential equations.

68.1 The Laplace transform of $e^{at} f(t)$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \qquad (1)$$

Thus $\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st}(e^{at} f(t))\,dt$

$$= \int_0^\infty e^{-(s-a)} f(t)\,dt \qquad (2)$$

(where a is a real constant)

68.2 Laplace transforms of the form $e^{at} f(t)$

(i) $\mathcal{L}\{e^{at} t^n\}$

Since $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$ from (viii) of Tab

Problem 1. Determine (a) $\mathcal{L}\{2t^4 e^{3t}\}$ (b) $\mathcal{L}\{4e^{3t} \cos 5t\}$

(a) From (i) of Table 68.1,

$$\mathcal{L}\{2t^4 e^{3t}\} = 2\mathcal{L}\{t^4 e^{3t}\} = 2\left(\frac{4!}{(s-3)^{4+1}}\right)$$

$$= \frac{2(4)(3)(2)}{(s-3)^5} = \frac{\mathbf{48}}{\mathbf{(s-3)^5}}$$

then $\mathcal{L}\{e^{at} t^n\} = \dfrac{n!}{(s-a)^{n+1}}$ from equation (2) above (provided $s > a$)

(ii) $\mathcal{L}\{e^{at} \sin \omega t\}$

Since $\mathcal{L}\{\sin \omega t\} = \dfrac{\omega}{s^2 + \omega^2}$ from (iv) of Table 67.1, page 728.

then $\mathcal{L}\{e^{at} \sin \omega t\} = \dfrac{\omega}{(s-a)^2 + \omega^2}$ from equation (2) (provided $s > a$)

(iii) $\mathcal{L}\{e^{at} \cosh \omega t\}$

Since $\mathcal{L}\{\cosh \omega t\} = \dfrac{s}{s^2 - \omega^2}$ from (ix) of Table 67.1, page 728.

then $\mathcal{L}\{e^{at} \cosh \omega t\} = \dfrac{s-a}{(s-a)^2 - \omega^2}$ from equation (2) (provided $s > a$)

68.2 Laplace transforms of the form $e^{at} f(t)$

(b) From (iii) of Table 68.1,

$$\mathcal{L}\{4e^{3t}\cos 5t\} = 4\mathcal{L}\{e^{3t}\cos 5t\}$$

$$= 4\left(\frac{s-3}{(s-3)^2+5^2}\right)$$

$$= \frac{4(s-3)}{s^2-6s+9+25}$$

$$= \boldsymbol{\frac{4(s-3)}{s^2-6s+34}}$$

then $\mathcal{L}\{e^{at}\ t^n\} = \dfrac{n!}{(s-a)^{n+1}}$ from equation (2) above (provided $s > a$)

(ii) $\mathcal{L}\{e^{at}\sin\omega t\}$

Since $\mathcal{L}\{\sin\omega t\} = \dfrac{\omega}{s^2+\omega^2}$ from (iv) of Table 67.1, page 728.

then $\mathcal{L}\{e^{at}\sin\omega t\} = \dfrac{\omega}{(s-a)^2+\omega^2}$ from equation (2) (provided $s > a$)

(iii) $\mathcal{L}\{e^{at}\cosh\omega t\}$

Since $\mathcal{L}\{\cosh\omega t\} = \dfrac{s}{s^2-\omega^2}$ from (ix) of Table 67.1, page 728.

then $\mathcal{L}\{e^{at}\cosh\omega t\} = \dfrac{s-a}{(s-a)^2-\omega^2}$ from equation (2) (provided $s > a$)

68.2 Laplace transforms of the form $e^{at} f(t)$

Function $e^{at} f(t)$ (a is a real constant)	Laplace transform $\mathcal{L}\{e^{at} f(t)\}$
(i) $e^{at} t^n$	$\dfrac{n!}{(s-a)^{n+1}}$
(ii) $e^{at} \sin \omega t$	$\dfrac{\omega}{(s-a)^2+\omega^2}$
(iii) $e^{at} \cos \omega t$	$\dfrac{s-a}{(s-a)^2+\omega^2}$
(iv) $e^{at} \sinh \omega t$	$\dfrac{\omega}{(s-a)^2-\omega^2}$
(v) $e^{at} \cosh \omega t$	$\dfrac{s-a}{(s-a)^2-\omega^2}$

68.2 Laplace transforms of the form $e^{at} f(t)$

Problem 2. Determine (a) $\mathcal{L}\{e^{-2t}\sin 3t\}$ (b) $\mathcal{L}\{3e^{\theta}\cosh 4\theta\}$

(a) From (ii) of Table 68.1,

$$\mathcal{L}\{e^{-2t}\sin 3t\} = \frac{3}{(s-(-2))^2+3^2} = \frac{3}{(s+2)^2+9}$$

$$= \frac{3}{s^2+4s+4+9} = \mathbf{\frac{3}{s^2+4s+13}}$$

(b) From (v) of Table 68.1,

$$\mathcal{L}\{3e^{\theta}\cosh 4\theta\} = 3\mathcal{L}\{e^{\theta}\cosh 4\theta\} = \frac{3(s-1)}{(s-1)^2-4^2}$$

$$= \frac{3(s-1)}{s^2-2s+1-16} = \mathbf{\frac{3(s-1)}{s^2-2s-15}}$$

68.2 Laplace transforms of the form $e^{at} f(t)$

Problem 3. Determine the Laplace transforms of (a) $5e^{-3t}\sinh 2t$ (b) $2e^{3t}(4\cos 2t - 5\sin 2t)$

(a) From (iv) of Table 68.1,

$$\mathcal{L}\{5e^{-3t}\sinh 2t\} = 5\mathcal{L}\{e^{-3t}\sinh 2t\}$$

$$= 5\left(\frac{2}{(s-(-3))^2 - 2^2}\right)$$

$$= \frac{10}{(s+3)^2 - 2^2} = \frac{10}{s^2+6s+9-4}$$

$$= \frac{\mathbf{10}}{\mathbf{s^2+6s+5}}$$

(b) $\mathcal{L}\{2e^{3t}(4\cos 2t - 5\sin 2t)\}$

$$= 8\mathcal{L}\{e^{3t}\cos 2t\} - 10\mathcal{L}\{e^{3t}\sin 2t\}$$

$$= \frac{8(s-3)}{(s-3)^2+2^2} - \frac{10(2)}{(s-3)^2+2^2}$$

from (iii) and (ii) of Table 68.1

$$= \frac{8(s-3) - 10(2)}{(s-3)^2+2^2} = \frac{\mathbf{8s-44}}{\mathbf{s^2-6s+13}}$$

68.2 Laplace transforms of the form $e^{at} f(t)$

1. (a) $2te^{2t}$ (b) t^2e^t
2. (a) $4t^3e^{-2t}$ (b) $\frac{1}{2}t^4e^{-3t}$
3. (a) $e^t \cos t$ (b) $3e^{2t}\sin 2t$
4. (a) $5e^{-2t}\cos 3t$ (b) $4e^{-5t}\sin t$
5. (a) $2e^t \sin^2 t$ (b) $\frac{1}{2}e^{3t}\cos^2 t$
6. (a) $e^t \sinh t$ (b) $3e^{2t}\cosh 4t$
7. (a) $2e^{-t}\sinh 3t$ (b) $\frac{1}{4}e^{-3t}\cosh 2t$
8. (a) $2e^t(\cos 3t - 3\sin 3t)$

   (b) $3e^{-2t}(\sinh 2t - 2\cosh 2t)$

68.3 The Laplace transforms of derivatives

(a) First derivative

Let the first derivative of $f(t)$ be $f'(t)$ then, from equation (1),

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

From Chapter 45, when integrating by parts

$$\int u\frac{dv}{dt}\,dt = uv - \int v\frac{du}{dt}\,dt$$

When evaluating $\int_0^\infty e^{-st} f'(t)\,dt$,

$$\text{let } u = e^{-st} \text{ and } \frac{dv}{dt} = f'(t)$$

from which,

$$\frac{du}{dt} = -se^{-st} \text{ and } v = \int f'(t)\,dt = f(t)$$

Hence $\int_0^\infty e^{-st} f'(t)\,dt$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty f(t)(-se^{-st})\,dt$$

$$= [0 - f(0)] + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\mathcal{L}\{f(t)\}$$

assuming $e^{-st} f(t) \to 0$ as $t \to \infty$, and $f(0)$ is the value of $f(t)$ at $t=0$. Hence,

$$\left.\begin{aligned} &\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) \\ \text{or}\quad &\mathcal{L}\left\{\frac{dy}{dx}\right\} = s\mathcal{L}\{y\} - y(0) \end{aligned}\right\} \qquad (3)$$

where $y(0)$ is the value of y at $x=0$

68.3 The Laplace transforms of derivatives

(b) Second derivative

Let the second derivative of $f(t)$ be $f''(t)$, then from equation (1),

$$\mathcal{L}\{f''(t)\} = \int_0^{\infty} e^{-st} f''(t)\,dt$$

Integrating by parts gives:

$$\int_0^{\infty} e^{-st} f''(t)\,dt = \left[e^{-st} f'(t)\right]_0^{\infty} + s\int_0^{\infty} e^{-st} f'(t)\,dt$$

$$= [0 - f'(0)] + s\mathcal{L}\{f'(t)\}$$

assuming $e^{-st} f'(t) \to 0$ as $t \to \infty$, and $f'(0)$ is the value of $f'(t)$ at $t=0$. Hence $\{f''(t)\} = -f'(0) + s[s(f(t)) - f(0)]$, from equation (3),

i.e.

$$\left.\begin{aligned} &\mathcal{L}\{f''(t)\} \\ &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) \\ \text{or}\quad &\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} \\ &= s^2\mathcal{L}\{y\} - sy(0) - y'(0) \end{aligned}\right\} \quad (4)$$

where $y'(0)$ is the value of $\frac{dy}{dx}$ at $x=0$

(b) Second derivative

Let the second derivative of $f(t)$ be $f''(t)$, then from equation (1),

$$\mathcal{L}\{f''(t)\} = \int_0^{\infty} e^{-st} f''(t)\,dt$$

Integrating by parts gives:

$$\int_0^{\infty} e^{-st} f''(t)\,dt = \left[e^{-st} f'(t)\right]_0^{\infty} + s\int_0^{\infty} e^{-st} f'(t)\,dt$$

$$= [0 - f'(0)] + s\mathcal{L}\{f'(t)\}$$

assuming $e^{-st} f'(t) \to 0$ as $t \to \infty$, and $f'(0)$ is the value of $f'(t)$ at $t=0$. Hence $\{f''(t)\} = -f'(0) + s[s(f(t)) - f(0)]$, from equation (3),

i.e.

$$\left.\begin{aligned} &\mathcal{L}\{f''(t)\} \\ &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0) \\ \text{or}\quad &\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} \\ &= s^2\mathcal{L}\{y\} - sy(0) - y'(0) \end{aligned}\right\} \qquad (4)$$

where $y'(0)$ is the value of $\frac{dy}{dx}$ at $x=0$

Problem 5. Use the Laplace transform of the first derivative to derive:

(a) $\mathcal{L}\{k\} = \frac{k}{s}$ (b) $\mathcal{L}\{2t\} = \frac{2}{s^2}$

From equation (3), $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$

(a) Let $f(t)=k$, then $f'(t)=0$ and $f(0)=k$

Substituting into equation (3) gives:

$$\mathcal{L}\{0\} = s\mathcal{L}\{k\} - k$$

i.e. $\quad k = s\mathcal{L}\{k\}$

Hence $\quad \boldsymbol{\mathcal{L}\{k\} = \frac{k}{s}}$

(b) Let $f(t)=2t$ then $f'(t)=2$ and $f(0)=0$

Substituting into equation (3) gives:

$$\mathcal{L}\{2\} = s\mathcal{L}\{2t\} - 0$$

i.e. $\quad \frac{2}{s} = s\mathcal{L}\{2t\}$

Hence $\quad \boldsymbol{\mathcal{L}\{2t\} = \frac{2}{s^2}}$

68.3 The Laplace transforms of derivatives

(c) $\mathcal{L}\{e^{-at}\} = \dfrac{1}{s+a}$

(c) Let $f(t)=e^{-at}$ then $f'(t)=-ae^{-at}$ and $f(0)=1$

Substituting into equation (3) gives:

$$\mathcal{L}\{-ae^{-at}\} = s\mathcal{L}\{e^{-at}\} - 1$$

$$-a\mathcal{L}\{e^{-at}\} = s\mathcal{L}\{e^{-at}\} - 1$$

$$1 = s\mathcal{L}\{e^{-at}\} + a\mathcal{L}\{e^{-at}\}$$

$$1 = (s+a)\mathcal{L}\{e^{-at}\}$$

Hence $\boldsymbol{\mathcal{L}\{e^{-at}\} = \dfrac{1}{s+a}}$

68.3 The Laplace transforms of derivatives

(c) $\mathcal{L}\{e^{-at}\} = \dfrac{1}{s+a}$

(c) Let $f(t) = e^{-at}$ then $f'(t) = -ae^{-at}$ and $f(0) = 1$

Substituting into equation (3) gives:

$$\mathcal{L}\{-ae^{-at}\} = s\mathcal{L}\{e^{-at}\} - 1$$

$$-a\mathcal{L}\{e^{-at}\} = s\mathcal{L}\{e^{-at}\} - 1$$

$$1 = s\mathcal{L}\{e^{-at}\} + a\mathcal{L}\{e^{-at}\}$$

$$1 = (s+a)\mathcal{L}\{e^{-at}\}$$

Hence $\boldsymbol{\mathcal{L}\{e^{-at}\} = \dfrac{1}{s+a}}$

68.3 The Laplace transforms of derivatives

Problem 6. Use the Laplace transform of the second derivative to derive

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$$

From equation (4),

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

Let $f(t) = \cos at$, then $f'(t) = -a\sin at$ and $f''(t) = -a^2\cos at$, $f(0) = 1$ and $f'(0) = 0$

Substituting into equation (4) gives:

$$\mathcal{L}\{-a^2\cos at\} = s^2\{\cos at\} - s(1) - 0$$

i.e. $$-a^2\mathcal{L}\{\cos at\} = s^2\mathcal{L}\{\cos at\} - s$$

Hence $$s = (s^2 + a^2)\mathcal{L}\{\cos at\}$$

from which, $$\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$$

68.3 The Laplace transforms of derivatives

1. Derive the Laplace transform of the first derivative from the definition of a Laplace transform. Hence derive the transform

$$\mathcal{L}\{1\} = \frac{1}{s}$$

2. Use the Laplace transform of the first derivative to derive the transforms:

(a) $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ (b) $\mathcal{L}\{3t^2\} = \frac{6}{s^3}$

3. Derive the Laplace transform of the second derivative from the definition of a Laplace transform. Hence derive the transform

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

4. Use the Laplace transform of the second derivative to derive the transforms:

(a) $\mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$

(b) $\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$

68.4 The initial and final value theorems

(a) The initial value theorem states:

$$\lim_{t\to 0} [f(t)] = \lim_{s\to\infty} [s\mathcal{L}\{f(t)\}]$$

For example, if $f(t) = 3e^{4t}$ then

$$\mathcal{L}\{3e^{4t}\} = \frac{3}{s-4}$$

By the initial value theorem,

$$\lim_{t\to 0} [3e^{4t}] = \lim_{s\to\infty} \left[s\left(\frac{3}{s-4}\right)\right]$$

i.e. $\quad 3e^0 = \infty\left(\frac{3}{\infty-4}\right)$

i.e. $\quad \mathbf{3 = 3}$, which illustrates the theorem.

68.4 The initial and final value theorems

(b) The final value theorem states:

$$\lim_{t\to\infty}[f(t)] = \lim_{s\to 0}[s\mathcal{L}\{f(t)\}]$$

For example, if $f(t) = 3e^{-4t}$ then:

$$\lim_{t\to\infty}[3e^{-4t}] = \lim_{s\to 0}\left[s\left(\frac{3}{s+4}\right)\right]$$

i.e. $$3e^{-\infty} = (0)\left(\frac{3}{0+4}\right)$$

i.e. $\mathbf{0 = 0}$, which illustrates the theorem.

LAPLACE TRANSFORMS

Inverse Laplace transforms

Introduction

Why it is important to understand: **Inverse Laplace transforms**

- **Laplace transforms and their inverses are a mathematical technique which allows us to solve differential equations, by primarily using algebraic methods. This simplification in the solving of equations, coupled with the ability to directly implement electrical components in their transformed form, makes the use of Laplace transforms widespread in both electrical engineering and control systems engineering.**

Introduction

Why it is important to understand: **Inverse Laplace transforms**

- **Laplace transforms have many further applications in mathematics, physics, optics, signal processing, and probability.**
- **This chapter specifically explains how the inverse Laplace transformis determined, which can also involve the use of partial fractions. In addition, poles and zeros of transfer functions are briefly explained; these are of importance in stability and control systems.**

At the end of this chapter, you should be able to:

- define the inverse Laplace transform
- use a standard list to determine the inverse Laplace transforms of simple functions
- determine inverse Laplace transforms using partial fractions
- define a pole and a zero
- determine poles and zeros for transfer functions, showing them on a pole–zero diagram

69.1 Definition of the inverse Laplace transform

If the Laplace transform of a function $f(t)$ is $F(s)$, i.e. $\mathcal{L}\{f(t)\} = F(s)$, then $f(t)$ is called the **inverse Laplace transform** of $F(s)$ and is written as $f(t) = \mathcal{L}^{-1}\{F(s)\}$

For example, since $\mathcal{L}\{1\} = \frac{1}{s}$ then $\boldsymbol{\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1}$

Similarly, since $\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$ then

$$\boldsymbol{\mathcal{L}^{-1}\left\{\frac{a}{s^2 + a^2}\right\} = \sin at}, \text{ and so on.}$$

69.2 Inverse Laplace transforms of simple functions

	$F(s)=\mathcal{L}\{f(t)\}$	$\mathcal{L}^{-1}\{F(s)\}=f(t)$
(i)	$\frac{1}{s}$	1
(ii)	$\frac{k}{s}$	k
(iii)	$\frac{1}{s-a}$	e^{at}
(iv)	$\frac{a}{s^2+a^2}$	$\sin at$
(v)	$\frac{s}{s^2+a^2}$	$\cos at$
(vi)	$\frac{1}{s^2}$	t
(vii)	$\frac{2!}{s^3}$	t^2
(viii)	$\frac{n!}{s^{n+1}}$	t^n
(ix)	$\frac{a}{s^2-a^2}$	$\sinh at$
(x)	$\frac{s}{s^2-a^2}$	$\cosh at$
(xi)	$\frac{n!}{(s-a)^{n+1}}$	$e^{at}t^n$
(xii)	$\frac{\omega}{(s-a)^2+\omega^2}$	$e^{at}\sin\omega t$
(xiii)	$\frac{s-a}{(s-a)^2+\omega^2}$	$e^{at}\cos\omega t$
(xiv)	$\frac{\omega}{(s-a)^2-\omega^2}$	$e^{at}\sinh\omega t$
(xv)	$\frac{s-a}{(s-a)^2-\omega^2}$	$e^{at}\cosh\omega t$

69.2 Inverse Laplace transforms of simple functions

Problem 1. Find the following inverse Laplace transforms:

(a) $\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}$ (b) $\mathcal{L}^{-1}\left\{\frac{5}{3s-1}\right\}$

(a)

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\}=\sin at,$$

Hence $\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\}$

$$=\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$=\mathbf{\frac{1}{3}\sin 3t}$$

(b) $\mathcal{L}^{-1}\left\{\frac{5}{3s-1}\right\}=\mathcal{L}^{-1}\left\{\frac{5}{3\left(s-\frac{1}{3}\right)}\right\}$

$$=\frac{5}{3}\mathcal{L}^{-1}\left\{\frac{1}{\left(s-\frac{1}{3}\right)}\right\}=\mathbf{\frac{5}{3}e^{\frac{1}{3}t}}$$

69.2 Inverse Laplace transforms of simple functions

Problem 2. Find the following inverse Laplace transforms:

(a) $\mathcal{L}^{-1}\left\{\frac{6}{s^3}\right\}$ (b) $\mathcal{L}^{-1}\left\{\frac{3}{s^4}\right\}$

(a) From (vii) of Table 69.1, $\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}=t^2$

Hence $\mathcal{L}^{-1}\left\{\frac{6}{s^3}\right\}=3\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\}=\mathbf{3t^2}$

(b) From (viii) of Table 69.1, if s is to have a power of 4 then $n=3$

Thus $\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\}=t^3$ i.e. $\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\}=t^3$

Hence $\mathcal{L}^{-1}\left\{\frac{3}{s^4}\right\}=\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\}=\mathbf{\frac{1}{2}t^3}$

69.2 Inverse Laplace transforms of simple functions

Problem 3. Determine

(a) $\mathcal{L}^{-1}\left\{\dfrac{7s}{s^2+4}\right\}$ (b) $\mathcal{L}^{-1}\left\{\dfrac{4s}{s^2-16}\right\}$

(a) $\mathcal{L}^{-1}\left\{\dfrac{7s}{s^2+4}\right\}=7\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2^2}\right\}=\mathbf{7\cos 2}\boldsymbol{t}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{4s}{s^2-16}\right\}=4\mathcal{L}^{-1}\left\{\dfrac{s}{s^2-4^2}\right\}$

$=\mathbf{4\cosh 4}\boldsymbol{t}$

69.2 Inverse Laplace transforms of simple functions

Problem 4. Find

(a) $\mathcal{L}^{-1}\left\{\frac{3}{s^2-7}\right\}$ (b) $\mathcal{L}^{-1}\left\{\frac{2}{(s-3)^5}\right\}$

(a)

$$\mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{3}{s^2-7}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s^2-(\sqrt{7})^2}\right\}$$

$$= \frac{3}{\sqrt{7}}\mathcal{L}^{-1}\left\{\frac{\sqrt{7}}{s^2-(\sqrt{7})^2}\right\}$$

$$= \frac{3}{\sqrt{7}}\sinh\sqrt{7}t$$

(b)

$$\mathcal{L}^{-1}\left\{\frac{n!}{(s-a)^{n+1}}\right\} = e^{at}t^n$$

Thus $\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^{n+1}}\right\} = \frac{1}{n!}e^{at}t^n$

and comparing with $\mathcal{L}^{-1}\left\{\frac{2}{(s-3)^5}\right\}$ shows that $n=4$ and $a=3$

Hence

$$\mathcal{L}^{-1}\left\{\frac{2}{(s-3)^5}\right\} = 2\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^5}\right\}$$

$$= 2\left(\frac{1}{4!}e^{3t}t^4\right) = \frac{1}{12}e^{3t}t^4$$

69.2 Inverse Laplace transforms of simple functions

Problem 5. Determine

(a) $\mathcal{L}^{-1}\left\{\dfrac{3}{s^2-4s+13}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{2(s+1)}{s^2+2s+10}\right\}$

(a) $$\mathcal{L}^{-1}\left\{\frac{3}{s^2-4s+13}\right\}=\mathcal{L}^{-1}\left\{\frac{3}{(s-2)^2+3^2}\right\}$$
$$=\mathbf{e^{2t}\sin 3t}$$

(b) $$\mathcal{L}^{-1}\left\{\frac{2(s+1)}{s^2+2s+10}\right\}=\mathcal{L}^{-1}\left\{\frac{2(s+1)}{(s+1)^2+3^2}\right\}$$
$$=\mathbf{2e^{-t}\cos 3t}$$

69.2 Inverse Laplace transforms of simple functions

Problem 6. Determine

(a) $\mathcal{L}^{-1}\left\{\dfrac{5}{s^2+2s-3}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{4s-3}{s^2-4s-5}\right\}$

(a) $$\mathcal{L}^{-1}\left\{\frac{5}{s^2+2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{5}{(s+1)^2-2^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{\frac{5}{2}(2)}{(s+1)^2-2^2}\right\}$$

$$= \frac{5}{2}e^{-t}\sinh 2t$$

(b) $$\mathcal{L}^{-1}\left\{\frac{4s-3}{s^2-4s-5}\right\} = \mathcal{L}^{-1}\left\{\frac{4s-3}{(s-2)^2-3^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{4(s-2)+5}{(s-2)^2-3^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{4(s-2)}{(s-2)^2-3^2}\right\} + \mathcal{L}^{-1}\left\{\frac{5}{(s-2)^2-3^2}\right\}$$

$$= 4e^{2t}\cosh 3t + \mathcal{L}^{-1}\left\{\frac{\frac{5}{3}(3)}{(s-2)^2-3^2}\right\}$$

$$= 4e^{2t}\cosh 3t + \frac{5}{3}e^{2t}\sinh 3t$$

69.2 Inverse Laplace transforms of simple functions

1. (a) $\frac{7}{s}$ (b) $\frac{2}{s-5}$

2. (a) $\frac{3}{2s+1}$ (b) $\frac{2s}{s^2+4}$

3. (a) $\frac{1}{s^2+25}$ (b) $\frac{4}{s^2+9}$

4. (a) $\frac{5s}{2s^2+18}$ (b) $\frac{6}{s^2}$

5. (a) $\frac{5}{s^3}$ (b) $\frac{8}{s^4}$

6. (a) $\frac{3s}{\frac{1}{2}s^2-8}$ (b) $\frac{7}{s^2-16}$

7. (a) $\frac{15}{3s^2-27}$ (b) $\frac{4}{(s-1)^3}$

8. (a) $\frac{1}{(s+2)^4}$ (b) $\frac{3}{(s-3)^5}$

9. (a) $\frac{s+1}{s^2+2s+10}$ (b) $\frac{3}{s^2+6s+13}$

10. (a) $\frac{2(s-3)}{s^2-6s+13}$ (b) $\frac{7}{s^2-8s+12}$

11. (a) $\frac{2s+5}{s^2+4s-5}$ (b) $\frac{3s+2}{s^2-8s+25}$

69.3 Inverse Laplace transforms using partial fractions

Problem 7. Determine $\mathcal{L}^{-1}\left\{\frac{4s-5}{s^2-s-2}\right\}$

$$\frac{4s-5}{s^2-s-2} \equiv \frac{4s-5}{(s-2)(s+1)} \equiv \frac{A}{(s-2)} + \frac{B}{(s+1)}$$

$$\equiv \frac{A(s+1)+B(s-2)}{(s-2)(s+1)}$$

Hence $4s-5 \equiv A(s+1)+B(s-2)$

When $s=2,\ 3=3A,$ from which, $A=1$

When $s=-1,\ -9=-3B,$ from which, $B=3$

Hence $\mathcal{L}^{-1}\left\{\frac{4s-5}{s^2-s-2}\right\}$

$$\equiv \mathcal{L}^{-1}\left\{\frac{1}{s-2} + \frac{3}{s+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{s+1}\right\}$$

$$= e^{2t} + 3e^{-t}$$

69.3 Inverse Laplace transforms using partial fractions

Problem 8. Find $\mathcal{L}^{-1}\left\{\frac{3s^3+s^2+12s+2}{(s-3)(s+1)^3}\right\}$

$$\frac{3s^3+s^2+12s+2}{(s-3)(s+1)^3}$$

$$\equiv \frac{A}{s-3}+\frac{B}{s+1}+\frac{C}{(s+1)^2}+\frac{D}{(s+1)^3}$$

$$\equiv \frac{\begin{pmatrix} A(s+1)^3+B(s-3)(s+1)^2 \\ +C(s-3)(s+1)+D(s-3)\end{pmatrix}}{(s-3)(s+1)^3}$$

Hence

$$3s^3+s^2+12s+2 \equiv A(s+1)^3+B(s-3)(s+1)^2 + C(s-3)(s+1)+D(s-3)$$

When $s=3$, $128=64A$, from which, $A=2$

When $s=-1$, $-12=-4D$, from which, $D=3$

Equating s^3 terms gives: $3=A+B$ from which, $B=1$

Equating constant terms gives:

$$2=A-3B-3C-3D$$

i.e. $$2=2-3-3C-9$$

from which, $3C=-12$ and $C=-4$

Hence

$$\mathcal{L}^{-1}\left\{\frac{3s^3+s^2+12s+2}{(s-3)(s+1)^3}\right\}$$

$$\equiv \mathcal{L}^{-1}\left\{\frac{2}{s-3}+\frac{1}{s+1}-\frac{4}{(s+1)^2}+\frac{3}{(s+1)^3}\right\}$$

$$= 2e^{3t}+e^{-t}-4e^{-t}t+\frac{3}{2}e^{-t}t^2$$

69.3 Inverse Laplace transforms using partial fractions

Problem 9. Determine

$$\mathcal{L}^{-1}\left\{\frac{5s^2+8s-1}{(s+3)(s^2+1)}\right\}$$

$$\frac{5s^2+8s-1}{(s+3)(s^2+1)} \equiv \frac{A}{s+3}+\frac{Bs+C}{(s^2+1)}$$

$$\equiv \frac{A(s^2+1)+(Bs+C)(s+3)}{(s+3)(s^2+1)}$$

Hence $5s^2+8s-1 \equiv A(s^2+1)+(Bs+C)(s+3)$

When $s=-3$, $20=10A$, from which, $A=2$

Equating s^2 terms gives: $5=A+B$, from which, $B=3$, since $A=2$

Equating s terms gives: $8=3B+C$, from which, $C=-1$, since $B=3$

Hence $\mathcal{L}^{-1}\left\{\frac{5s^2+8s-1}{(s+3)(s^2+1)}\right\}$

$$\equiv \mathcal{L}^{-1}\left\{\frac{2}{s+3}+\frac{3s-1}{s^2+1}\right\}$$

$$\equiv \mathcal{L}^{-1}\left\{\frac{2}{s+3}\right\}+\mathcal{L}^{-1}\left\{\frac{3s}{s^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= 2e^{-3t}+3\cos t - \sin t$$

69.3 Inverse Laplace transforms using partial fractions

Problem 10. Find $\mathcal{L}^{-1}\left\{\dfrac{7s+13}{s(s^2+4s+13)}\right\}$

$$\frac{7s+13}{s(s^2+4s+13)} \equiv \frac{A}{s} + \frac{Bs+C}{s^2+4s+13}$$

$$\equiv \frac{A(s^2+4s+13)+(Bs+C)(s)}{s(s^2+4s+13)}$$

Hence $7s+13 \equiv A(s^2+4s+13)+(Bs+C)(s)$.

When $s=0$, $13=13A$, from which, $A=1$

Equating s^2 terms gives: $0=A+B$, from which, $B=-1$

Equating s terms gives: $7=4A+C$, from which, $C=3$

Hence $\mathcal{L}^{-1}\left\{\dfrac{7s+13}{s(s^2+4s+13)}\right\}$

$$\equiv \mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{-s+3}{s^2+4s+13}\right\}$$

$$\equiv \mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{-s+3}{s^2+4s+13}\right\}$$

$$\equiv \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\mathcal{L}^{-1}\left\{\frac{-s+3}{(s+2)^2+3^2}\right\}$$

$$\equiv \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\mathcal{L}^{-1}\left\{\frac{-(s+2)+5}{(s+2)^2+3^2}\right\}$$

$$\equiv \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}-\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\}$$

$$+\mathcal{L}^{-1}\left\{\frac{5}{(s+2)^2+3^2}\right\}$$

$$\equiv \mathbf{1-e^{-2t}\cos 3t+\frac{5}{3}e^{-2t}\sin 3t}$$

69.3 Inverse Laplace transforms using partial fractions

1. $\dfrac{11-3s}{s^2+2s-3}$

2. $\dfrac{2s^2-9s-35}{(s+1)(s-2)(s+3)}$

3. $\dfrac{5s^2-2s-19}{(s+3)(s-1)^2}$

4. $\dfrac{3s^2+16s+15}{(s+3)^3}$

5. $\dfrac{7s^2+5s+13}{(s^2+2)(s+1)}$

6. $\dfrac{3+6s+4s^2-2s^3}{s^2(s^2+3)}$

7. $\dfrac{26-s^2}{s(s^2+4s+13)}$

LAPLACE TRANSFORMS

The Laplace transform of the Heaviside function

Introduction

Why it is important to understand: **The Laplace transform of the Heaviside function**

- **The Heaviside unit step function is used in the mathematics of control theory and signal processing to represent a signal that switches on at a specified time and stays switched on indefinitely. It is also used in structural mechanics to describe different types of structural loads.**
- **The Heaviside function has applications in engineering where periodic functions are represented. In many physical situations things change suddenly; brakes are applied, a switch is thrown, collisions occur. The Heaviside unit function is very useful for representing sudden change.**

At the end of this chapter, you should be able to:

- define the Heaviside unit step function

- use a standard list to determine the Laplace transform of $H(t-c)$

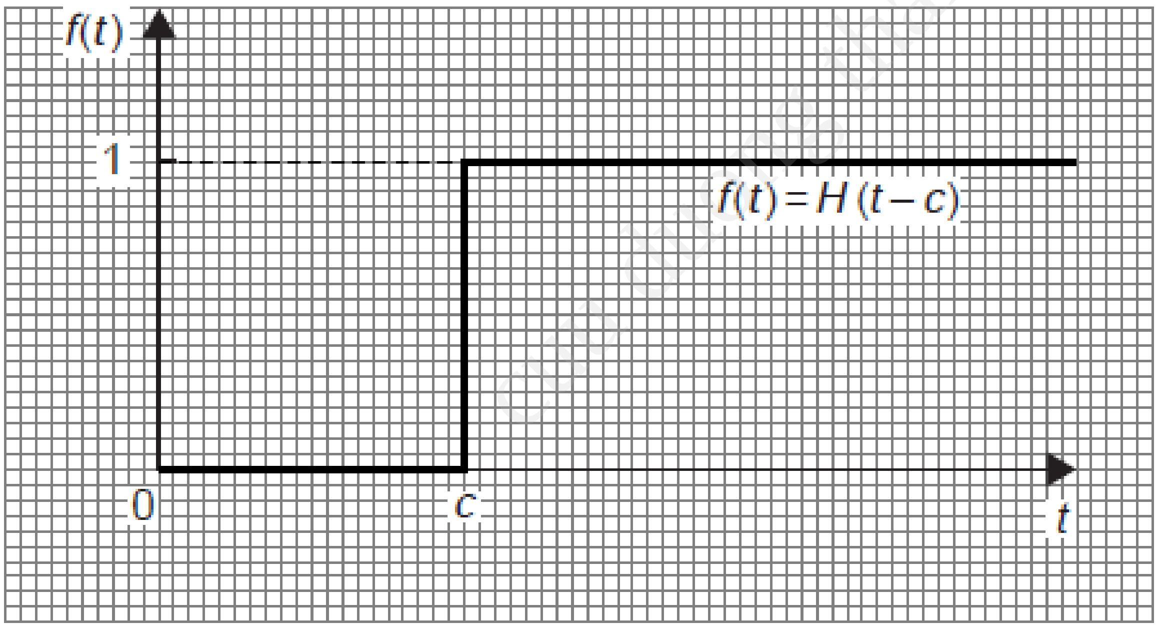
- use a standard list to determine the Laplace transform of $H(t-c) \cdot f(t-c)$

- determine the inverse transforms of Heaviside functions

70.1 Heaviside unit step function

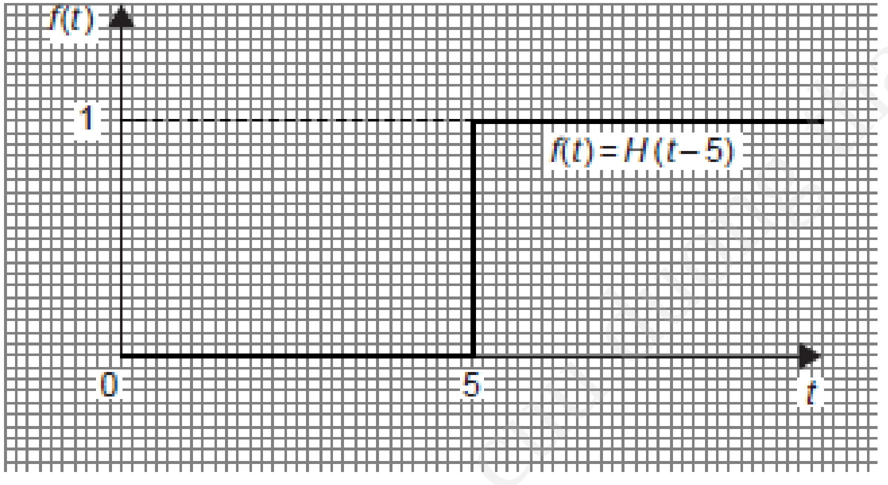
The switching process can be described mathematically by the function called the **Unit Step Function** – otherwise known as the **Heaviside unit step function**.

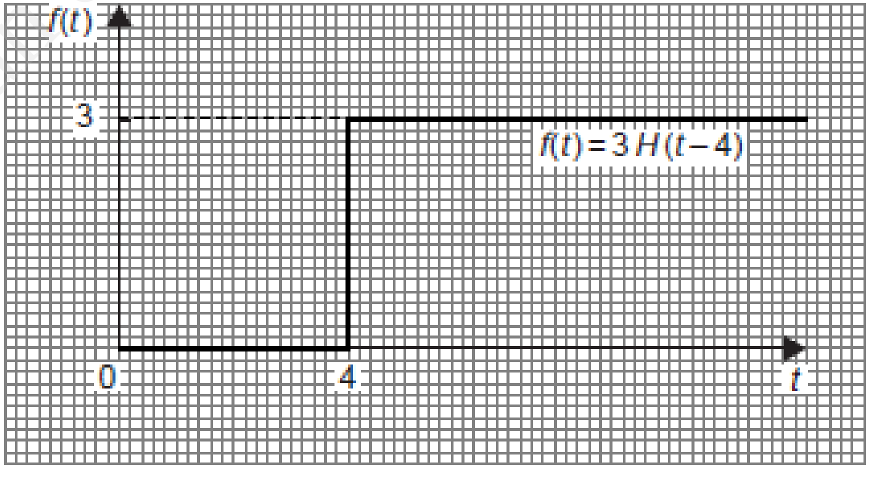
$$f(t) = H(t-c) \quad \text{or} \quad u(t-c)$$



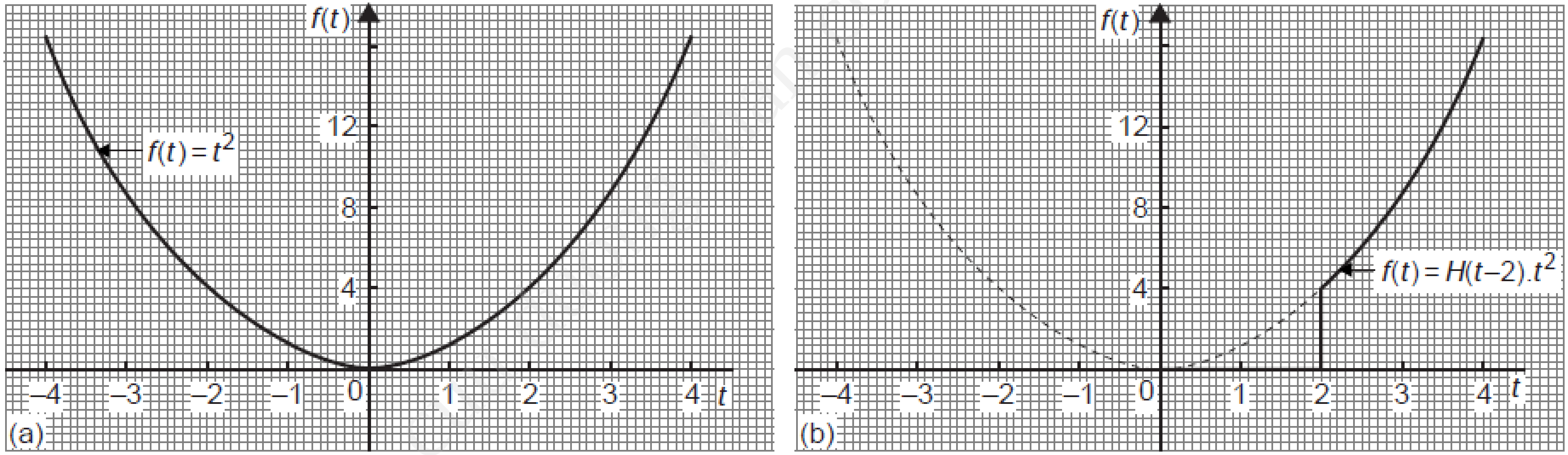
70.1 Heaviside unit step function

$$f(t) = H(t-c) \quad \text{or} \quad u(t-c)$$



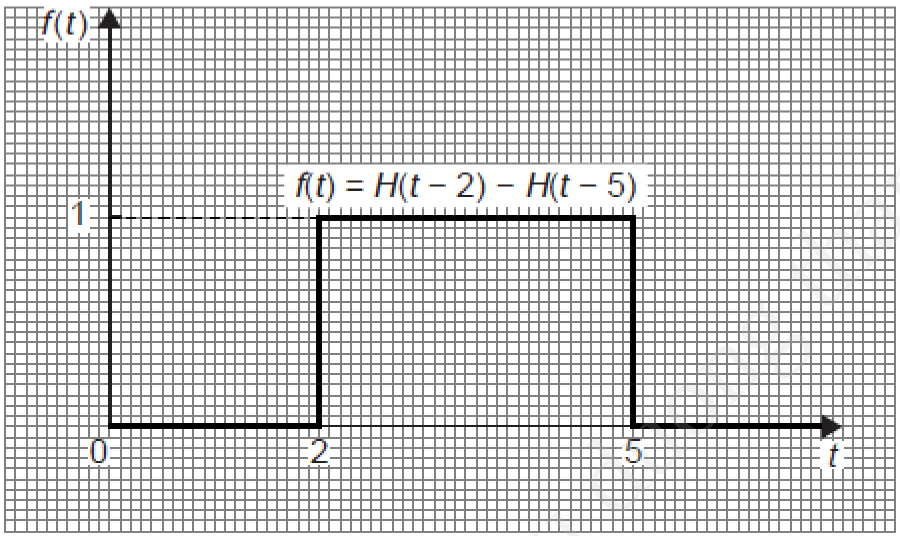


70.1 Heaviside unit step function



70.1 Heaviside unit step function

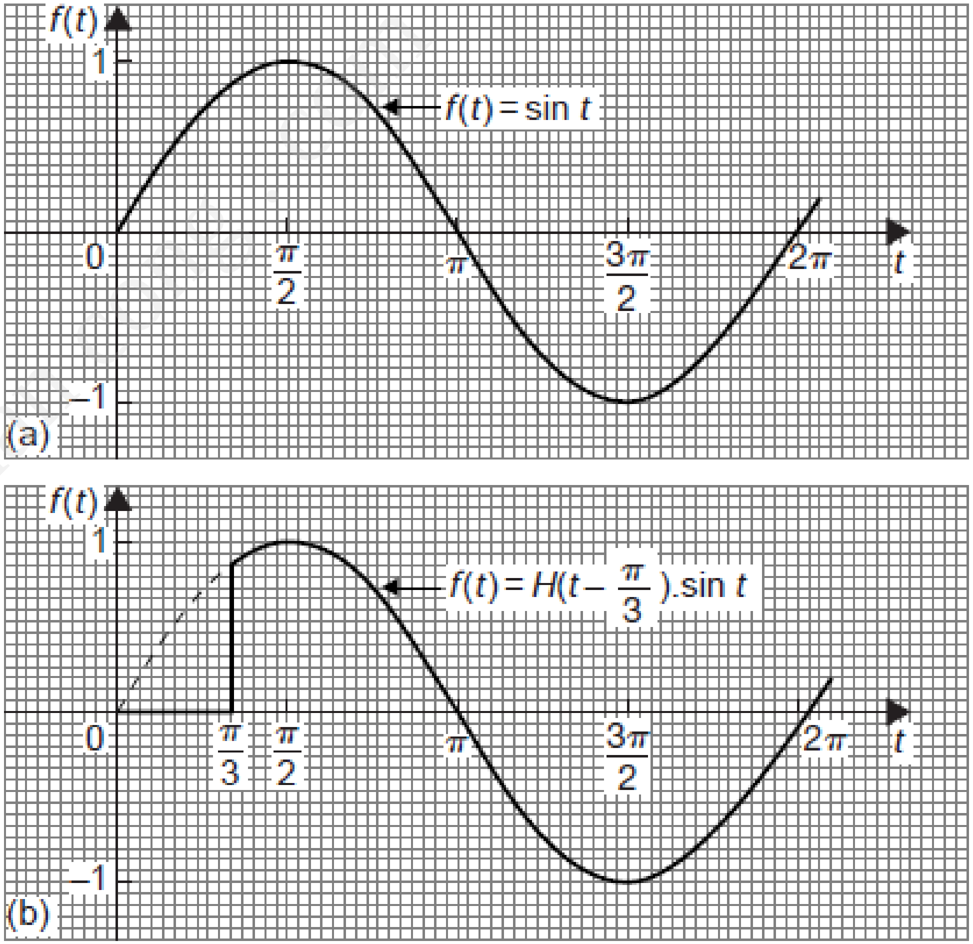
$$V(t) = H(t-a) - H(t-b)$$



70.1 Heaviside unit step function

Problem 4. Sketch the graph of
$f(t) = H(t - \pi/3).\sin t$

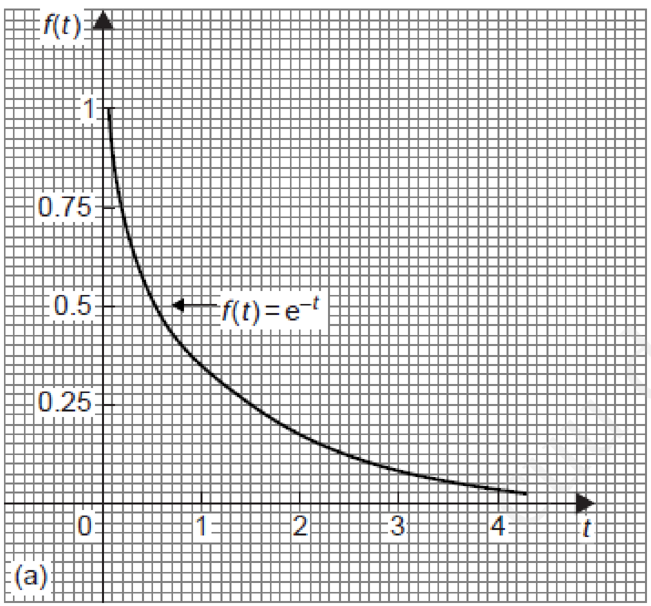
Fig. 70.10(a) shows a graph of $f(t) = \sin t$; the graph shown in Fig. 70.10(b) is $\boldsymbol{f(t) = H(t - \pi/3) \cdot \sin t}$ where the graph of sin t does not 'switch on' until $t = \pi/3$

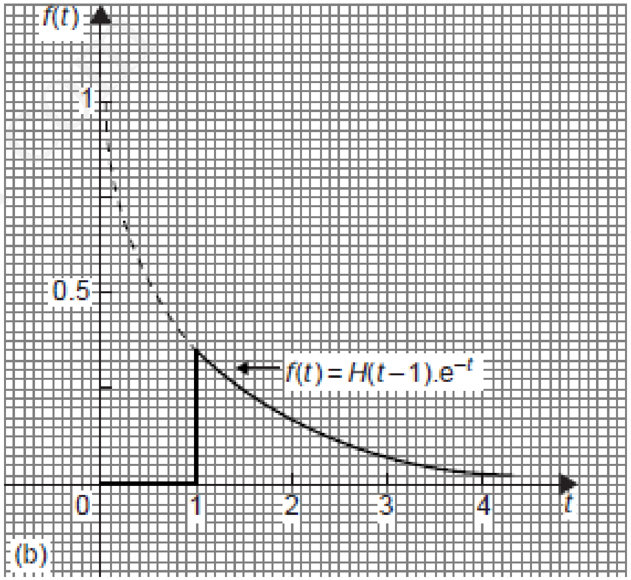


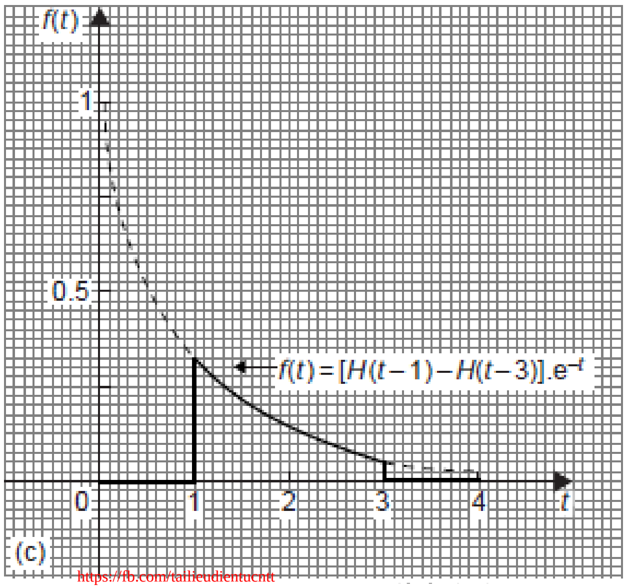
70.1 Heaviside unit step function

(a) $f(t) = H(t-1).e^{-t}$

(b) $f(t) = [H(t-1) - H(t-3)].e^{-t}$







70.1 Heaviside unit step function

1. A 6 V source is switched on at time $t = 4$ s. Write the function in terms of the Heaviside step function and sketch the waveform.

2. Write the function $V(t) = \begin{cases} 2 & \text{for } 0 < t < 5 \\ 0 & \text{for } t > 5 \end{cases}$ in terms of the Heaviside step function and sketch the waveform.

In problems 3 to 12, sketch graphs of the given functions.

3. $f(t) = H(t-2)$
4. $f(t) = H(t)$
5. $f(t) = 4H(t-1)$
6. $f(t) = 7\,H(t-5)$
7. $f(t) = H\left(t - \frac{\pi}{4}\right) \cdot \cos t$
8. $f(t) = 3H\left(t - \frac{\pi}{2}\right) \cdot \cos\left(t - \frac{\pi}{6}\right)$
9. $f(t) = H(t-1) \cdot t^2$
10. $f(t) = H(t-2) \cdot e^{-\frac{t}{2}}$
11. $f(t) = [H(t-2) - H(t-5)] \cdot e^{-\frac{t}{4}}$
12. $f(t) = 5H\left(t - \frac{\pi}{3}\right) \cdot \sin\left(t + \frac{\pi}{4}\right)$

70.2 Laplace transform of *H(t−c)*

From the definition of a Laplace transform,

$$\mathcal{L}\{H(t-c)\} = \int_0^\infty e^{-st} H(t-c)\,dt$$

However, $e^{-st}H(t-c) = \begin{cases} 0 & \text{for } 0 < t < c \\ e^{-st} & \text{for } t \geq c \end{cases}$

Hence, $\mathcal{L}\{H(t-c)\} = \displaystyle\int_0^\infty e^{-st} H(t-c)\,dt$

$$= \int_c^\infty e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_c^\infty = \left[\frac{e^{-s(\infty)}}{-s} - \frac{e^{-sc}}{-s}\right]$$

$$= \left[0 - \frac{e^{-sc}}{-s}\right] = \frac{e^{-sc}}{s}$$

When $c = 0$ (i.e. a unit step at the origin), $\mathcal{L}\{H(t)\} = \dfrac{e^{-s(0)}}{s} = \dfrac{1}{s}$

Summarising, $\mathcal{L}\{H(t)\} = \dfrac{1}{s}$ and $\mathcal{L}\{H(t-c)\} = \dfrac{e^{-cs}}{s}$

From the definition of $H(t)$: $\mathcal{L}\{1\} = \{1 \cdot H(t)\}$

$$\mathcal{L}\{t\} = \{t \cdot H(t)\}$$

and $\qquad \mathcal{L}\{f(t)\} = \{f(t) \cdot H(t)\}$

70.3 Laplace transform of *H(t−c) · f(t−c)*

It may be shown that:

$$\mathcal{L}\{H(t-c)\cdot f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s)$$

where $F(s) = \mathcal{L}\{f(t)\}$

Problem 7. Determine $\mathcal{L}\{4H(t-5)\}$

From above, $\mathcal{L}\{H(t-c)\cdot f(t-c)\} = e^{-cs}F(s)$ where in this case, $F(s) = \mathcal{L}\{4\}$ and $c = 5$

Hence, $\mathcal{L}\{4H(t-5)\} = e^{-5s}\left(\frac{4}{s}\right)$ from (ii) of Table 67.1, page 728

$$= \frac{4e^{-5s}}{s}$$

Problem 8. Determine $\mathcal{L}\{H(t-3)\cdot(t-3)^2\}$

From above, $\mathcal{L}\{H(t-c)\cdot f(t-c)\} = e^{-cs}F(s)$ where in this case, $F(s) = \mathcal{L}\{t^2\}$ and $c = 3$

Note that $F(s)$ is the transform of t^2 and not of $(t-3)^2$

Hence, $\mathcal{L}\{H(t-3)\cdot f(t-3)^2\} = e^{-3s}\left(\frac{2!}{s^3}\right)$ from (vii) of Table 67.1, page 728

$$= \frac{2e^{-3s}}{s^3}$$

70.3 Laplace transform of *H(t−c) · f(t−c)*

It may be shown that:

$$\mathcal{L}\{H(t-c)\cdot f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s)$$

where $F(s) = \mathcal{L}\{f(t)\}$

Problem 9. Determine $\mathcal{L}\{H(t-2).\sin(t-2)\}$

From above, $\mathcal{L}\{H(t-c)\cdot f(t-c)\} = e^{-cs}F(s)$ where in this case, $F(s) = \mathcal{L}\{\sin t\}$ and $c = 2$

Hence, $\mathcal{L}\{H(t-2)\cdot\sin(t-2)\} = e^{-2s}\left(\frac{1}{s^2+1^2}\right)$

from (iv) of Table 67.1, page 728

$$= \frac{\mathbf{e^{-2s}}}{\mathbf{s^2+1}}$$

Problem 10. Determine $\mathcal{L}\{H(t-1)\cdot\sin 4(t-1)\}$

From above, $\mathcal{L}\{H(t-c)\cdot f(t-c)\} = e^{-cs}F(s)$ where in this case, $F(s) = \mathcal{L}\{\sin 4t\}$ and $c = 1$

Hence, $\mathcal{L}\{H(t-1)\cdot\sin 4(t-1)\} = e^{-s}\left(\frac{4}{s^2+4^2}\right)$

from (iv) of Table 67.1, page 728

$$= \frac{\mathbf{4\,e^{-s}}}{\mathbf{s^2+16}}$$

70.3 Laplace transform of *H(t−c) · f(t−c)*

It may be shown that:

$$\mathcal{L}\{H(t-c)\cdot f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s)$$

where $F(s) = \mathcal{L}\{f(t)\}$

Problem 11. Determine $\mathcal{L}\{H(t-3)\cdot e^{t-3}\}$

From above, $\mathcal{L}\{H(t-c)\cdot f(t-c)\} = e^{-cs}F(s)$ where in this case, $F(s) = \mathcal{L}\{e^t\}$ and $c = 3$

Hence, $\mathcal{L}\{H(t-3)\cdot e^{t-3}\} = e^{-3s}\left(\frac{1}{s-1}\right)$ from (iii) of Table 67.1, page 728

$$= \frac{\mathbf{e^{-3s}}}{\mathbf{s-1}}$$

Problem 12. Determine

$$\mathcal{L}\left\{H\left(t-\frac{\pi}{2}\right)\cdot\cos 3\left(t-\frac{\pi}{2}\right)\right\}$$

From above, $\mathcal{L}\{H(t-c)\cdot f(t-c)\} = e^{-cs}F(s)$ where in this case, $F(s) = \mathcal{L}\{\cos 3t\}$ and $c = \frac{\pi}{2}$

Hence, $\mathcal{L}\left\{\left(t-\frac{\pi}{2}\right)\cdot\cos 3\left(1-\frac{\pi}{2}\right)\right\} =$

$e^{-\frac{\pi}{2}s}\left(\frac{s}{s^2+3^2}\right)$ from (v) of Table 67.1, page 728

$$= \frac{\mathbf{s e^{-\frac{\pi}{2}s}}}{\mathbf{s^2+9}}$$

70.3 Laplace transform of *H(t−c) · f(t−c)*

1. Determine $\mathcal{L}\{H(t-1)\}$
2. Determine $\mathcal{L}\{7H(t-3)\}$
3. Determine $\mathcal{L}\{H(t-2)\cdot(t-2)^2\}$
4. Determine $\mathcal{L}\{H(t-3)\cdot\sin(t-3)\}$
5. Determine $\mathcal{L}\{H(t-4)\cdot e^{t-4}\}$
6. Determine $\mathcal{L}\{H(t-5)\cdot\sin 3(t-5)\}$
7. Determine $\mathcal{L}\{H(t-1)\cdot(t-1)^3\}$
8. Determine $\mathcal{L}\{H(t-6)\cdot\cos 3(t-6)\}$
9. Determine $\mathcal{L}\{5H(t-5)\cdot\sinh 2(t-5)\}$
10. Determine $\mathcal{L}\{H\left(t-\frac{\pi}{3}\right)\cdot\cos 2\left(t-\frac{\pi}{3}\right)\}$
11. Determine $\mathcal{L}\{2H(t-3)\cdot e^{t-3}\}$
12. Determine $\mathcal{L}\{3H(t-2)\cdot\cosh(t-2)\}$

70.3 Laplace transform of *H(t−c) · f(t−c)*

1. Determine $\mathcal{L}\{H(t-1)\}$
2. Determine $\mathcal{L}\{7H(t-3)\}$
3. Determine $\mathcal{L}\{H(t-2)\cdot(t-2)^2\}$
4. Determine $\mathcal{L}\{H(t-3)\cdot\sin(t-3)\}$
5. Determine $\mathcal{L}\{H(t-4)\cdot e^{t-4}\}$
6. Determine $\mathcal{L}\{H(t-5)\cdot\sin 3(t-5)\}$
7. Determine $\mathcal{L}\{H(t-1)\cdot(t-1)^3\}$
8. Determine $\mathcal{L}\{H(t-6)\cdot\cos 3(t-6)\}$
9. Determine $\mathcal{L}\{5H(t-5)\cdot\sinh 2(t-5)\}$
10. Determine $\mathcal{L}\{H\left(t-\frac{\pi}{3}\right)\cdot\cos 2\left(t-\frac{\pi}{3}\right)\}$
11. Determine $\mathcal{L}\{2H(t-3)\cdot e^{t-3}\}$
12. Determine $\mathcal{L}\{3H(t-2)\cdot\cosh(t-2)\}$

70.4 Inverse Laplace transforms of Heaviside functions

if $F(s) = \mathcal{L}\{f(t)\}$, then $e^{-cs}F(s) = \mathcal{L}\{H(t-c)\cdot f(t-c)\}$

This is known as the **second shift theorem** and is used when finding **inverse Laplace transforms**, as demonstrated in the following worked problems.

Problem 13. Determine $\mathcal{L}^{-1}\left\{\dfrac{3e^{-2s}}{s}\right\}$

Part of the numerator corresponds to e^{-cs} where $c = 2$. This indicates $H(t-2)$

Then $\dfrac{3}{s} = F(s) = \mathcal{L}\{3\}$

page 728

Hence, $\mathcal{L}^{-1}\left\{\dfrac{3e^{-2s}}{s}\right\} = 3H(t-2)$

Problem 14. Determine the inverse of $\dfrac{e^{-3s}}{s^2}$

The numerator corresponds to e^{-cs} where $c = 3$. This indicates $H(t-3)$

$\dfrac{1}{s^2} = F(s) = \mathcal{L}\{t\}$

page 728

Then $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s^2}\right\} = H(t-3)\cdot(t-3)$

70.4 Inverse Laplace transforms of Heaviside functions

Problem 15. Determine $\mathcal{L}^{-1}\left\{\dfrac{8\,e^{-4s}}{s^2+4}\right\}$

Part of the numerator corresponds to e^{-cs} where $c = 4$. This indicates $H(t-4)$

$\dfrac{8}{s^2+4}$ may be written as: $4\left(\dfrac{2}{s^2+2^2}\right)$

Then $4\left(\dfrac{2}{s^2+2^2}\right) = F(s) = \mathcal{L}\{4\sin 2t\}$

Hence, $\mathcal{L}^{-1}\left\{\dfrac{8\,e^{-4s}}{s^2+4}\right\} = H(t-4)\cdot 4\sin 2(t-4)$

$$= 4H(t-4)\cdot \sin 2(t-4)$$

70.4 Inverse Laplace transforms of Heaviside functions

Problem 16. Determine $\mathcal{L}^{-1}\left\{\frac{5\,s\mathrm{e}^{-2\,s}}{s^2+9}\right\}$

Part of the numerator corresponds to e^{-cs} where $c = 2$.
This indicates $H(t-2)$

$\frac{5\,s}{s^2+9}$ may be written as: $5\left(\frac{s}{s^2+3^2}\right)$

Then $5\left(\frac{s}{s^2+3^2}\right) = F(s) = \mathcal{L}\{5\cos 3t\}$

Table 67.1, page 728

Hence, $\mathcal{L}^{-1}\left\{\frac{5\,s\mathrm{e}^{-2s}}{s^2+3^2}\right\} = H(t-2)\cdot 5\cos 3(t-2)$

$$= 5H(t-2)\cdot\cos 3(t-2)$$

70.4 Inverse Laplace transforms of Heaviside functions

Problem 17. Determine $\mathcal{L}^{-1}\left\{\dfrac{7\,e^{-3s}}{s^2-1}\right\}$

Part of the numerator corresponds to e^{-cs} where $c = 3$.
This indicates $H(t-3)$

$\dfrac{7}{s^2-1}$ may be written as: $7\left(\dfrac{1}{s^2-1^2}\right)$

Then $7\left(\dfrac{1}{s^2-1^2}\right) = F(s) = \mathcal{L}\{7\sinh t\}$

Hence, $\mathcal{L}^{-1}\left\{\dfrac{7\,e^{-3s}}{s^2-1}\right\} = H(t-3)\cdot 7\sinh(t-3)$

$$= 7H(t-3)\cdot \sinh(t-3)$$

70.4 Inverse Laplace transforms of Heaviside functions

1. Determine $\mathcal{L}^{-1}\left\{\frac{e^{-9s}}{s}\right\}$

2. Determine $\mathcal{L}^{-1}\left\{\frac{4e^{-3s}}{s}\right\}$

3. Determine $\mathcal{L}^{-1}\left\{\frac{2e^{-2s}}{s^2}\right\}$

4. Determine $\mathcal{L}^{-1}\left\{\frac{5e^{-2s}}{s^2+1}\right\}$

5. Determine $\mathcal{L}^{-1}\left\{\frac{3s\,e^{-4s}}{s^2+16}\right\}$

6. Determine $\mathcal{L}^{-1}\left\{\frac{6e^{-2s}}{s^2-1}\right\}$

7. Determine $\mathcal{L}^{-1}\left\{\frac{3e^{-6s}}{s^3}\right\}$

8. Determine $\mathcal{L}^{-1}\left\{\frac{2s\,e^{-4s}}{s^2-16}\right\}$

9. Determine $\mathcal{L}^{-1}\left\{\frac{2s\,e^{-\frac{1}{2}s}}{s^2+5}\right\}$

10. Determine $\mathcal{L}^{-1}\left\{\frac{4e^{-7s}}{s-1}\right\}$

LAPLACE TRANSFORMS

The Laplace transform of the Heaviside function

Introduction

Why it is important to understand: **The Laplace transform of the Heaviside function**

- **The Heaviside unit step function is used in the mathematics of control theory and signal processing to represent a signal that switches on at a specified time and stays switched on indefinitely. It is also used in structural mechanics to describe different types of structural loads.**
- **The Heaviside function has applications in engineering where periodic functions are represented. In many physical situations things change suddenly; brakes are applied, a switch is thrown, collisions occur. The Heaviside unit function is very useful for representing sudden change.**

At the end of this chapter, you should be able to:

- define the Heaviside unit step function

- use a standard list to determine the Laplace transform of $H(t - c)$

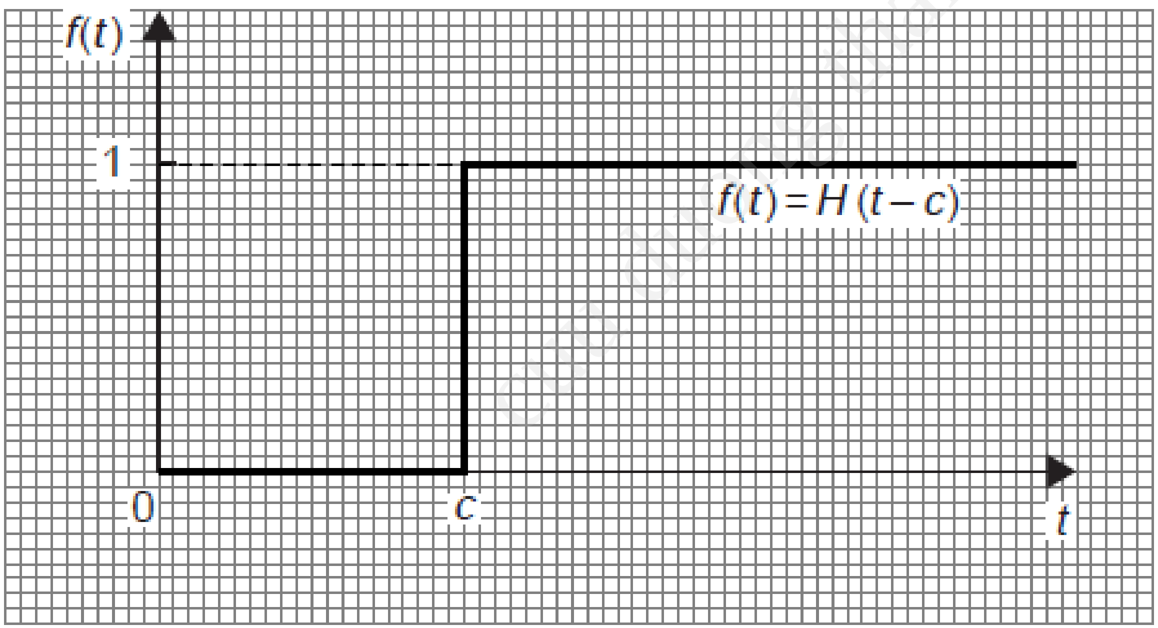
- use a standard list to determine the Laplace transform of $H(t - c) \cdot f(t - c)$

- determine the inverse transforms of Heaviside functions

70.1 Heaviside unit step function

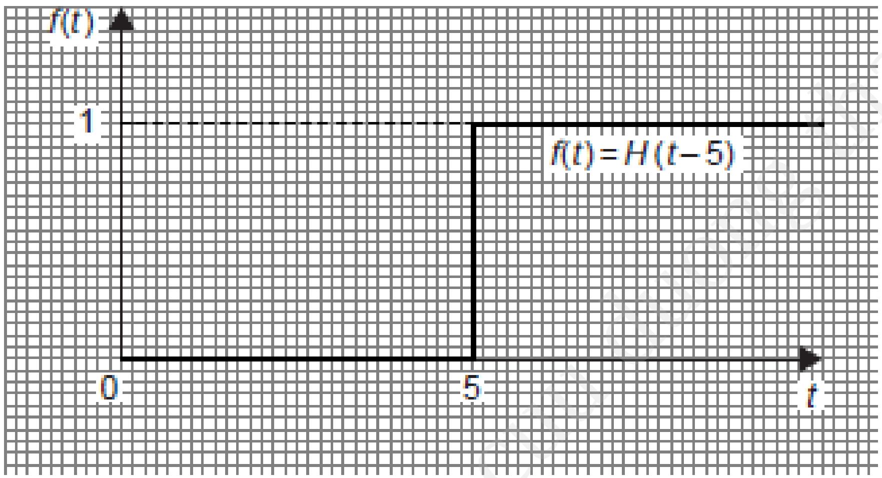
The switching process can be described mathematically by the function called the **Unit Step Function** – otherwise known as the **Heaviside unit step function**.

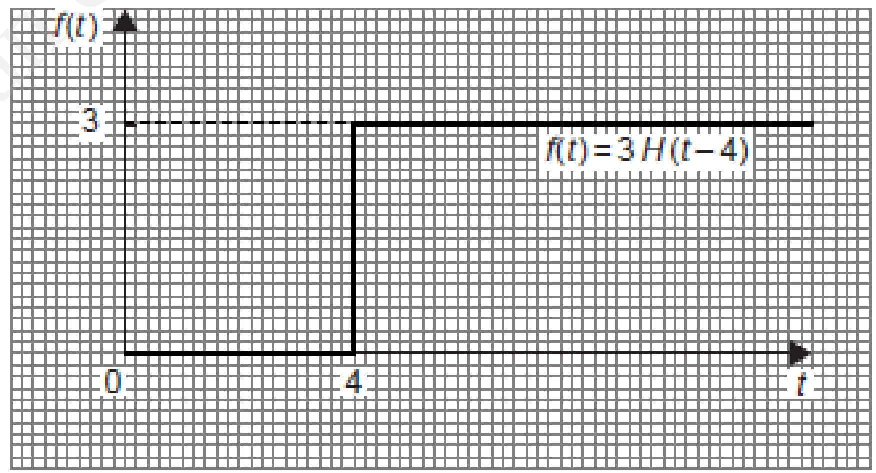
$$f(t) = H(t-c) \quad \text{or} \quad u(t-c)$$



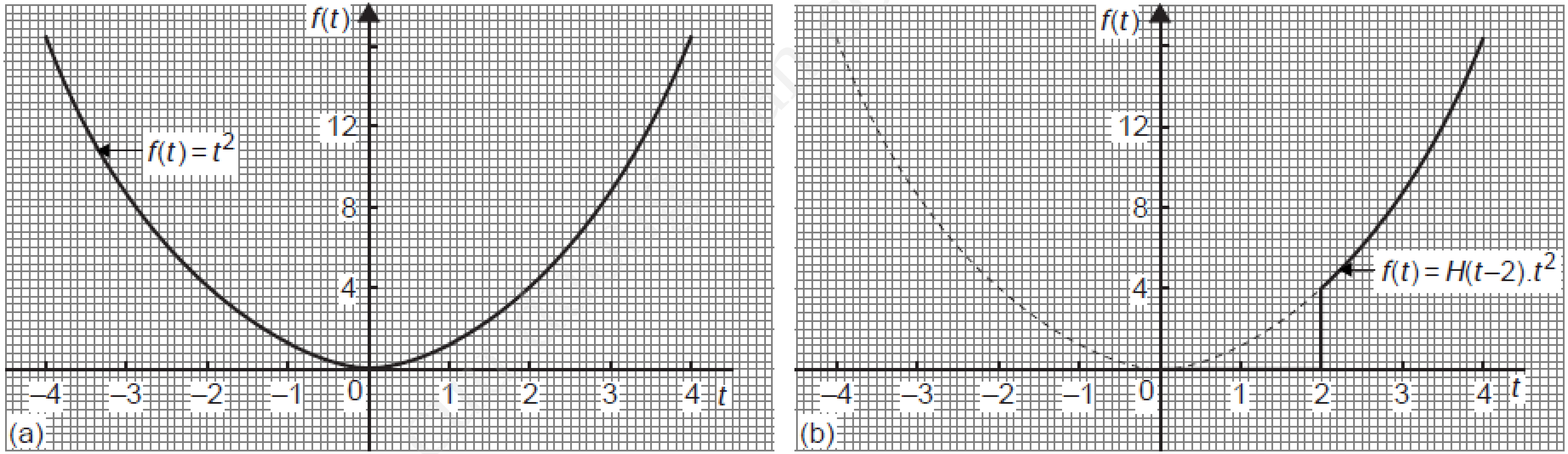
70.1 Heaviside unit step function

$$f(t) = H(t-c) \quad \text{or} \quad u(t-c)$$



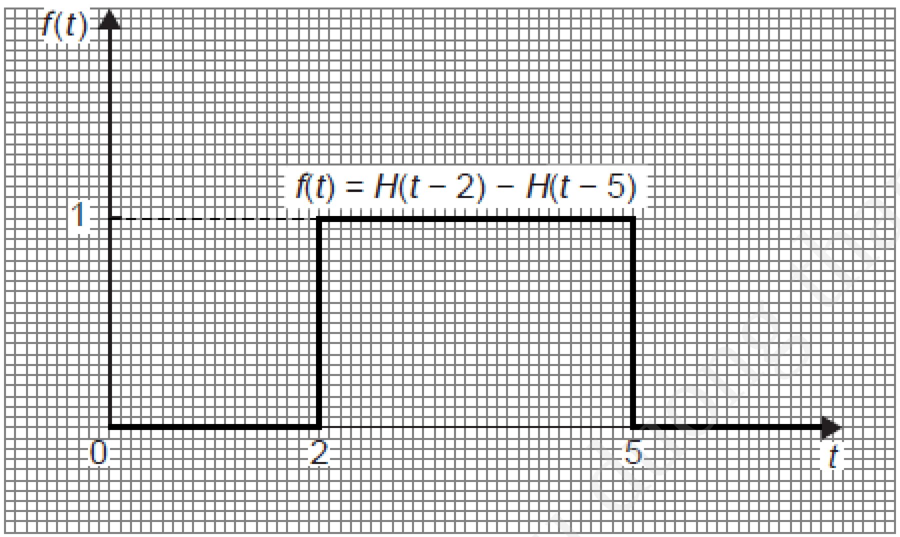


70.1 Heaviside unit step function



70.1 Heaviside unit step function

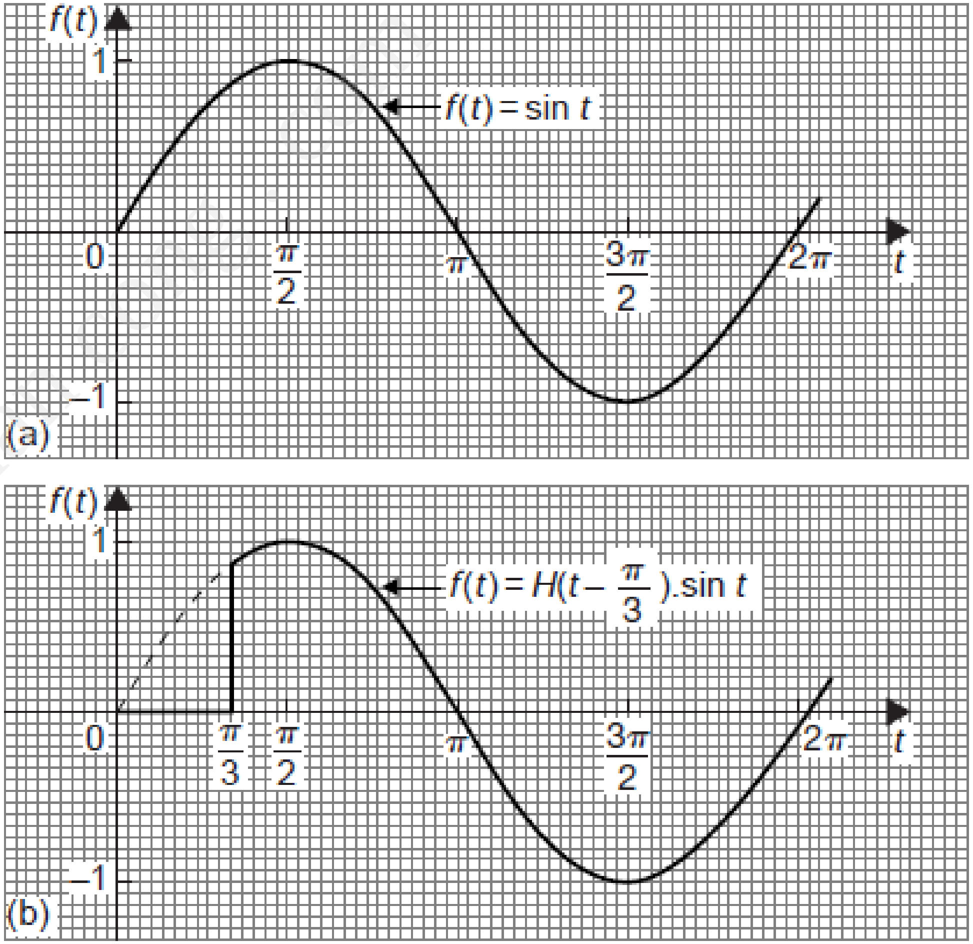
$$V(t) = H(t-a) - H(t-b)$$



70.1 Heaviside unit step function

Problem 4. Sketch the graph of
$$f(t) = H(t - \pi/3).\sin t$$

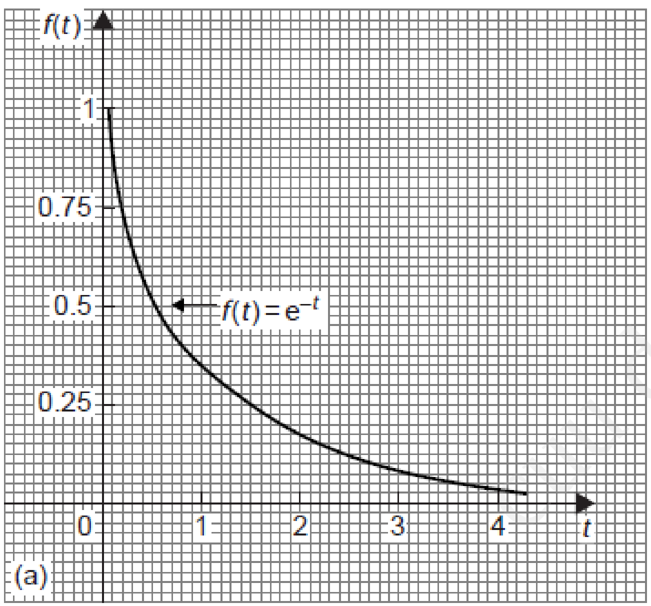
Fig. 70.10(a) shows a graph of $f(t) = \sin t$; the graph shown in Fig. 70.10(b) is $f(t) = H(t - \pi/3) \cdot \sin t$ where the graph of $\sin t$ does not 'switch on' until $t = \pi/3$

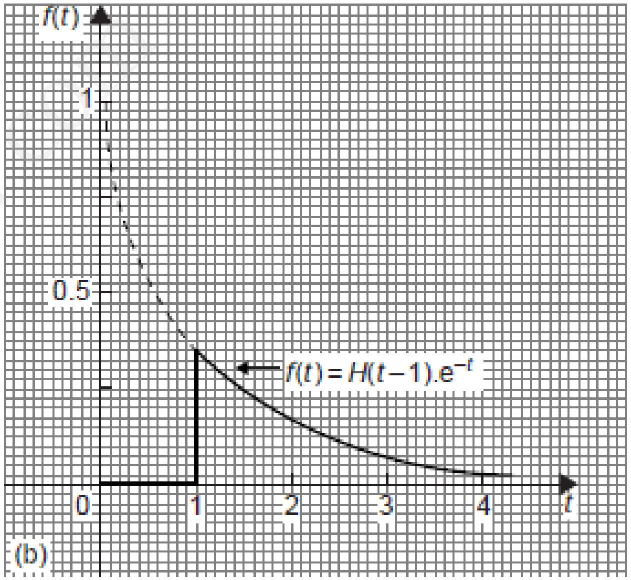


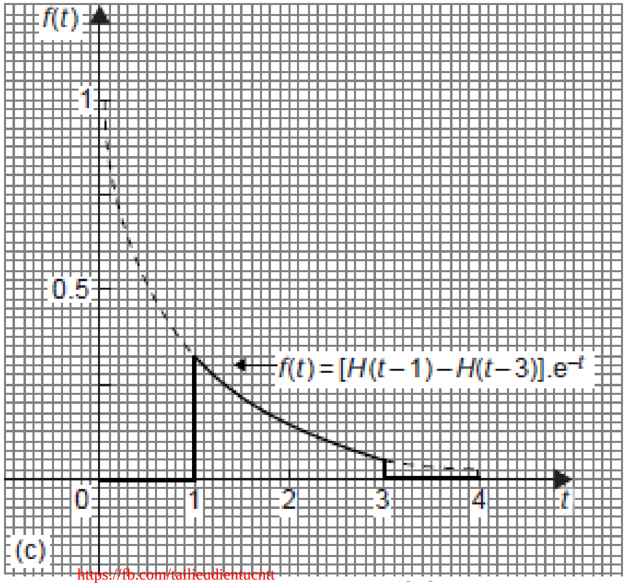
70.1 Heaviside unit step function

(a) $f(t) = H(t-1).e^{-t}$

(b) $f(t) = [H(t-1) - H(t-3)].e^{-t}$







70.1 Heaviside unit step function

1. A 6 V source is switched on at time $t = 4$ s. Write the function in terms of the Heaviside step function and sketch the waveform.

2. Write the function $V(t) = \begin{cases} 2 & \text{for } 0 < t < 5 \\ 0 & \text{for } t > 5 \end{cases}$ in terms of the Heaviside step function and sketch the waveform.

In problems 3 to 12, sketch graphs of the given functions.

3. $f(t) = H(t-2)$
4. $f(t) = H(t)$
5. $f(t) = 4H(t-1)$
6. $f(t) = 7H(t-5)$
7. $f(t) = H\left(t - \frac{\pi}{4}\right) \cdot \cos t$
8. $f(t) = 3H\left(t - \frac{\pi}{2}\right) \cdot \cos\left(t - \frac{\pi}{6}\right)$
9. $f(t) = H(t-1) \cdot t^2$
10. $f(t) = H(t-2) \cdot e^{-\frac{t}{2}}$
11. $f(t) = [H(t-2) - H(t-5)] \cdot e^{-\frac{t}{4}}$
12. $f(t) = 5H\left(t - \frac{\pi}{3}\right) \cdot \sin\left(t + \frac{\pi}{4}\right)$

70.2 Laplace transform of *H(t−c)*

From the definition of a Laplace transform,

$$\mathcal{L}\{H(t-c)\} = \int_0^{\infty} e^{-st} H(t-c)\,dt$$

However, $e^{-st} H(t-c) = \begin{cases} 0 & \text{for } 0 < t < c \\ e^{-st} & \text{for } t \geq c \end{cases}$

Hence, $\mathcal{L}\{H(t-c)\} = \int_0^{\infty} e^{-st} H(t-c)\,dt$

$$= \int_c^{\infty} e^{-st}\,dt = \left[\frac{e^{-st}}{-s}\right]_c^{\infty} = \left[\frac{e^{-s(\infty)}}{-s} - \frac{e^{-sc}}{-s}\right]$$

$$= \left[0 - \frac{e^{-sc}}{-s}\right] = \frac{e^{-sc}}{s}$$

When $c = 0$ (i.e. a unit step at the origin), $\mathcal{L}\{H(t)\} = \frac{e^{-s(0)}}{s} = \frac{1}{s}$

Summarising, $\mathcal{L}\{H(t)\} = \frac{1}{s}$ and $\mathcal{L}\{H(t-c)\} = \frac{e^{-cs}}{s}$

From the definition of $H(t)$: $\mathcal{L}\{1\} = \{1 \cdot H(t)\}$

$$\mathcal{L}\{t\} = \{t \cdot H(t)\}$$

and $\mathcal{L}\{f(t)\} = \{f(t) \cdot H(t)\}$

70.3 Laplace transform of *H(t−c) · f(t−c)*

It may be shown that:

$$\mathcal{L}\{H(t-c)\cdot f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s)$$

where $F(s) = \mathcal{L}\{f(t)\}$

Problem 7. Determine $\mathcal{L}\{4H(t-5)\}$

From above, $\mathcal{L}\{H(t-c)\cdot f(t-c)\} = e^{-cs}F(s)$ where in this case, $F(s) = \mathcal{L}\{4\}$ and $c = 5$

Hence, $\mathcal{L}\{4H(t-5)\} = e^{-5s}\left(\frac{4}{s}\right)$ from (ii) of Table 67.1, page 728

$$= \frac{4e^{-5s}}{s}$$

Problem 8. Determine $\mathcal{L}\{H(t-3)\cdot(t-3)^2\}$

From above, $\mathcal{L}\{H(t-c)\cdot f(t-c)\} = e^{-cs}F(s)$ where in this case, $F(s) = \mathcal{L}\{t^2\}$ and $c = 3$

Note that $F(s)$ is the transform of t^2 and not of $(t-3)^2$

Hence, $\mathcal{L}\{H(t-3)\cdot f(t-3)^2\} = e^{-3s}\left(\frac{2!}{s^3}\right)$ from (vii) of Table 67.1, page 728

$$= \frac{2e^{-3s}}{s^3}$$

70.3 Laplace transform of *H(t−c) · f(t−c)*

It may be shown that:

$$\mathcal{L}\{H(t-c)\cdot f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s)$$

where $F(s) = \mathcal{L}\{f(t)\}$

Problem 9. Determine $\mathcal{L}\{H(t-2).\sin(t-2)\}$

From above, $\mathcal{L}\{H(t-c)\cdot f(t-c)\} = e^{-cs}F(s)$ where in this case, $F(s) = \mathcal{L}\{\sin t\}$ and $c = 2$

Hence, $\mathcal{L}\{H(t-2)\cdot\sin(t-2)\} = e^{-2s}\left(\frac{1}{s^2+1^2}\right)$

from (iv) of Table 67.1, page 728

$$= \mathbf{\frac{e^{-2s}}{s^2+1}}$$

Problem 10. Determine $\mathcal{L}\{H(t-1)\cdot\sin 4(t-1)\}$

From above, $\mathcal{L}\{H(t-c)\cdot f(t-c)\} = e^{-cs}F(s)$ where in this case, $F(s) = \mathcal{L}\{\sin 4t\}$ and $c = 1$

Hence, $\mathcal{L}\{H(t-1)\cdot\sin 4(t-1)\} = e^{-s}\left(\frac{4}{s^2+4^2}\right)$

from (iv) of Table 67.1, page 728

$$= \mathbf{\frac{4e^{-s}}{s^2+16}}$$

70.3 Laplace transform of *H(t−c) · f(t−c)*

It may be shown that:

$$\mathcal{L}\{H(t-c)\cdot f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} = e^{-cs}F(s)$$

where $F(s) = \mathcal{L}\{f(t)\}$

Problem 11. Determine $\mathcal{L}\{H(t-3)\cdot e^{t-3}\}$

From above, $\mathcal{L}\{H(t-c)\cdot f(t-c)\} = e^{-cs}F(s)$ where in this case, $F(s) = \mathcal{L}\{e^t\}$ and $c = 3$

Hence, $\mathcal{L}\{H(t-3)\cdot e^{t-3}\} = e^{-3s}\left(\frac{1}{s-1}\right)$ from (iii) of Table 67.1, page 728

$$= \frac{e^{-3s}}{s-1}$$

Problem 12. Determine $\mathcal{L}\left\{H\left(t-\frac{\pi}{2}\right)\cdot\cos 3\left(t-\frac{\pi}{2}\right)\right\}$

From above, $\mathcal{L}\{H(t-c)\cdot f(t-c)\} = e^{-cs}F(s)$ where in this case, $F(s) = \mathcal{L}\{\cos 3t\}$ and $c = \frac{\pi}{2}$

Hence, $\mathcal{L}\left\{\left(t-\frac{\pi}{2}\right)\cdot\cos 3\left(1-\frac{\pi}{2}\right)\right\} =$ $e^{-\frac{\pi}{2}s}\left(\frac{s}{s^2+3^2}\right)$ from (v) of Table 67.1, page 728

$$= \frac{se^{-\frac{\pi}{2}s}}{s^2+9}$$

70.3 Laplace transform of *H(t−c) · f(t−c)*

1. Determine $\mathcal{L}\{H(t-1)\}$
2. Determine $\mathcal{L}\{7H(t-3)\}$
3. Determine $\mathcal{L}\{H(t-2)\cdot(t-2)^2\}$
4. Determine $\mathcal{L}\{H(t-3)\cdot\sin(t-3)\}$
5. Determine $\mathcal{L}\{H(t-4)\cdot \mathrm{e}^{t-4}\}$
6. Determine $\mathcal{L}\{H(t-5)\cdot\sin 3(t-5)\}$
7. Determine $\mathcal{L}\{H(t-1)\cdot(t-1)^3\}$
8. Determine $\mathcal{L}\{H(t-6)\cdot\cos 3(t-6)\}$
9. Determine $\mathcal{L}\{5H(t-5)\cdot\sinh 2(t-5)\}$
10. Determine $\mathcal{L}\{H\left(t-\frac{\pi}{3}\right)\cdot\cos 2\left(t-\frac{\pi}{3}\right)\}$
11. Determine $\mathcal{L}\{2H(t-3)\cdot \mathrm{e}^{t-3}\}$
12. Determine $\mathcal{L}\{3H(t-2)\cdot\cosh(t-2)\}$

70.3 Laplace transform of *H(t−c) · f(t−c)*

1. Determine $\mathcal{L}\{H(t-1)\}$
2. Determine $\mathcal{L}\{7H(t-3)\}$
3. Determine $\mathcal{L}\{H(t-2)\cdot(t-2)^2\}$
4. Determine $\mathcal{L}\{H(t-3)\cdot\sin(t-3)\}$
5. Determine $\mathcal{L}\{H(t-4)\cdot e^{t-4}\}$
6. Determine $\mathcal{L}\{H(t-5)\cdot\sin 3(t-5)\}$
7. Determine $\mathcal{L}\{H(t-1)\cdot(t-1)^3\}$
8. Determine $\mathcal{L}\{H(t-6)\cdot\cos 3(t-6)\}$
9. Determine $\mathcal{L}\{5H(t-5)\cdot\sinh 2(t-5)\}$
10. Determine $\mathcal{L}\{H\left(t-\frac{\pi}{3}\right)\cdot\cos 2\left(t-\frac{\pi}{3}\right)\}$
11. Determine $\mathcal{L}\{2H(t-3)\cdot e^{t-3}\}$
12. Determine $\mathcal{L}\{3H(t-2)\cdot\cosh(t-2)\}$

70.4 Inverse Laplace transforms of Heaviside functions

if $F(s) = \mathcal{L}\{f(t)\}$, then $e^{-cs}F(s) = \mathcal{L}\{H(t-c)\cdot f(t-c)\}$

This is known as the **second shift theorem** and is used when finding **inverse Laplace transforms**, as demonstrated in the following worked problems.

Problem 13. Determine $\mathcal{L}^{-1}\left\{\frac{3e^{-2s}}{s}\right\}$

Part of the numerator corresponds to e^{-cs} where $c = 2$ This indicates $H(t-2)$

Then $\frac{3}{s} = F(s) = \mathcal{L}\{3\}$

page 728

Hence, $\mathcal{L}^{-1}\left\{\frac{3e^{-2s}}{s}\right\} = 3H(t-2)$

Problem 14. Determine the inverse of $\frac{e^{-3s}}{s^2}$

The numerator corresponds to e^{-cs} where $c = 3$. This indicates $H(t-3)$

$\frac{1}{s^2} = F(s) = \mathcal{L}\{t\}$

page 728

Then $\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2}\right\} = H(t-3)\cdot(t-3)$

70.4 Inverse Laplace transforms of Heaviside functions

Problem 15. Determine $\mathcal{L}^{-1}\left\{\dfrac{8e^{-4s}}{s^2+4}\right\}$

Part of the numerator corresponds to e^{-cs} where $c = 4$. This indicates $H(t-4)$

$\dfrac{8}{s^2+4}$ may be written as: $4\left(\dfrac{2}{s^2+2^2}\right)$

Then $4\left(\dfrac{2}{s^2+2^2}\right) = F(s) = \mathcal{L}\{4\sin 2t\}$

Hence, $\mathcal{L}^{-1}\left\{\dfrac{8e^{-4s}}{s^2+4}\right\} = H(t-4)\cdot 4\sin 2(t-4)$

$$= 4H(t-4)\cdot \sin 2(t-4)$$

70.4 Inverse Laplace transforms of Heaviside functions

Problem 16. Determine $\mathcal{L}^{-1}\left\{\dfrac{5se^{-2s}}{s^2+9}\right\}$

Part of the numerator corresponds to e^{-cs} where $c = 2$.
This indicates $H(t-2)$

$\dfrac{5s}{s^2+9}$ may be written as: $5\left(\dfrac{s}{s^2+3^2}\right)$

Then $5\left(\dfrac{s}{s^2+3^2}\right) = F(s) = \mathcal{L}\{5\cos 3t\}$

Table 67.1, page 728

$$\text{Hence, } \mathcal{L}^{-1}\left\{\frac{5\,se^{-2s}}{s^2+3^2}\right\} = H(t-2)\cdot 5\cos 3(t-2)$$

$$= 5H(t-2)\cdot\cos 3(t-2)$$

70.4 Inverse Laplace transforms of Heaviside functions

Problem 17. Determine $\mathcal{L}^{-1}\left\{\dfrac{7\,e^{-3s}}{s^2-1}\right\}$

Part of the numerator corresponds to e^{-cs} where $c = 3$.
This indicates $H(t-3)$

$\dfrac{7}{s^2-1}$ may be written as: $7\left(\dfrac{1}{s^2-1^2}\right)$

Then $7\left(\dfrac{1}{s^2-1^2}\right) = F(s) = \mathcal{L}\{7\sinh t\}$

Hence, $\mathcal{L}^{-1}\left\{\dfrac{7\,e^{-3s}}{s^2-1}\right\} = H(t-3)\cdot 7\sinh(t-3)$

$$= 7H(t-3)\cdot\sinh(t-3)$$

70.4 Inverse Laplace transforms of Heaviside functions

1. Determine $\mathcal{L}^{-1}\left\{\frac{e^{-9s}}{s}\right\}$

2. Determine $\mathcal{L}^{-1}\left\{\frac{4e^{-3s}}{s}\right\}$

3. Determine $\mathcal{L}^{-1}\left\{\frac{2e^{-2s}}{s^2}\right\}$

4. Determine $\mathcal{L}^{-1}\left\{\frac{5e^{-2s}}{s^2+1}\right\}$

5. Determine $\mathcal{L}^{-1}\left\{\frac{3s\,e^{-4s}}{s^2+16}\right\}$

6. Determine $\mathcal{L}^{-1}\left\{\frac{6e^{-2s}}{s^2-1}\right\}$

7. Determine $\mathcal{L}^{-1}\left\{\frac{3e^{-6s}}{s^3}\right\}$

8. Determine $\mathcal{L}^{-1}\left\{\frac{2s\,e^{-4s}}{s^2-16}\right\}$

9. Determine $\mathcal{L}^{-1}\left\{\frac{2s\,e^{-\frac{1}{2}s}}{s^2+5}\right\}$

10. Determine $\mathcal{L}^{-1}\left\{\frac{4e^{-7s}}{s-1}\right\}$

LAPLACE TRANSFORMS

The solution of differential equations using Laplace transforms

Introduction

Why it is important to understand: **The solution of differential equations using Laplace transforms**

- **Laplace transforms and their inverses are a mathematical technique which allows us to solve differential equations, by primarily using algebraic methods.**
- **This simplification in the solving of equations, coupled with the ability to directly implement electrical components in their transformed form, makes the use of Laplace transforms widespread in both electrical engineering and control systems engineering.**
- **The procedures explained in previous chapters are used in this chapter which demonstrates how differential equations are solved using Laplace transforms.**

At the end of this chapter, you should be able to:

- understand the procedure to solve differential equations using Laplace transforms

- solve differential equations using Laplace transforms

71.1 Introduction

An alternative method of

solving differential equations

71.2 Procedure to solve differentia equations by using Laplace transforms

(i) Take the Laplace transform of both sides of the differential equation by applying the formulae for the Laplace transforms of derivatives

(ii) Put in the given initial conditions, i.e. $y(0)$ and $y'(0)$.

(iii) Rearrange the equation to make $\mathcal{L}\{y\}$ the subject.

(iv) Determine y by using, where necessary, partial fractions and taking the inverse of each term by using Laplace transforms Table

71.3 Worked problems on solving differential equations using Laplace transforms

Problem 1. Use Laplace transforms to solve the differential equation
$2\frac{d^2y}{dx^2}+5\frac{dy}{dx}-3y=0$, given that when $x=0$, $y=4$ and $\frac{dy}{dx}=9$

(i) $2\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\}+5\mathcal{L}\left\{\frac{dy}{dx}\right\}-3\mathcal{L}\{y\}=\mathcal{L}\{0\}$

$$2[s^2\mathcal{L}\{y\}-sy(0)-y'(0)]+5[s\mathcal{L}\{y\}-y(0)]-3\mathcal{L}\{y\}=0,$$

(ii) $y(0)=4$ and $y'(0)=9$

Thus $2[s^2\mathcal{L}\{y\}-4s-9]+5[s\mathcal{L}\{y\}-4]-3\mathcal{L}\{y\}=0$

i.e. $2s^2\mathcal{L}\{y\}-8s-18+5s\mathcal{L}\{y\}-20-3\mathcal{L}\{y\}=0$

(iii) Rearranging gives:

$$(2s^2+5s-3)\mathcal{L}\{y\}=8s+38$$

i.e. $\mathcal{L}\{y\}=\frac{8s+38}{2s^2+5s-3}$

(iv) $y=\mathcal{L}^{-1}\left\{\frac{8s+38}{2s^2+5s-3}\right\}$

$$\frac{8s+38}{2s^2+5s-3}\equiv\frac{8s+38}{(2s-1)(s+3)}$$

71.3 Worked problems on solving differential equations using Laplace transforms

(iii) Rearranging gives:

$$(2s^2+5s-3)\mathcal{L}\{y\}=8s+38$$

i.e. $\mathcal{L}\{y\}=\dfrac{8s+38}{2s^2+5s-3}$

(iv) $y=\mathcal{L}^{-1}\left\{\dfrac{8s+38}{2s^2+5s-3}\right\}$

$$\frac{8s+38}{2s^2+5s-3}\equiv\frac{8s+38}{(2s-1)(s+3)}$$

$$\equiv\frac{A}{2s-1}+\frac{B}{s+3}$$

$$\equiv\frac{A(s+3)+B(2s-1)}{(2s-1)(s+3)}$$

Hence $8s+38=A(s+3)+B(2s-1)$

When $s=\frac{1}{2}$, $42=3\frac{1}{2}A$, from which, $A=12$

When $s=-3$, $14=-7B$, from which, $B=-2$

Hence $y=\mathcal{L}^{-1}\left\{\dfrac{8s+38}{2s^2+5s-3}\right\}$

$$=\mathcal{L}^{-1}\left\{\frac{12}{2s-1}-\frac{2}{s+3}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{12}{2\left(s-\frac{1}{2}\right)}\right\}-\mathcal{L}^{-1}\left\{\frac{2}{s+3}\right\}$$

Hence $\boldsymbol{y=6e^{\frac{1}{2}x}-2e^{-3x}}$, from (iii) of

71.3 Worked problems on solving differential equations using Laplace transforms

Problem 2. Use Laplace transforms to solve the differential equation:

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+13y=0, \text{ given that when } x=0,\ y=3$$
and $\dfrac{dy}{dx}=7$

This is the same as Problem 3 of Chapter 53, page 555. Using the above procedure:

(i) $$\mathcal{L}\left\{\frac{d^2x}{dy^2}\right\}+6\mathcal{L}\left\{\frac{dy}{dx}\right\}+13\mathcal{L}\{y\}=\mathcal{L}\{0\}$$

Hence $[s^2\mathcal{L}\{y\}-sy(0)-y'(0)]$

$$+6[s\mathcal{L}\{y\}-y(0)]+13\mathcal{L}\{y\}=0,$$

from equations (3) and (4) of Chapter 68.

(ii) $y(0)=3$ and $y'(0)=7$

Thus $s^2\mathcal{L}\{y\}-3s-7+6s\mathcal{L}\{y\}$

$$-18+13\mathcal{L}\{y\}=0$$

(iii) Rearranging gives:

$$(s^2+6s+13)\mathcal{L}\{y\}=3s+25$$

i.e. $$\mathcal{L}\{y\}=\frac{3s+25}{s^2+6s+13}$$

(iv) $$y=\mathcal{L}^{-1}\left\{\frac{3s+25}{s^2+6s+13}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{3s+25}{(s+3)^2+2^2}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{3(s+3)+16}{(s+3)^2+2^2}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{3(s+3)}{(s+3)^2+2^2}\right\}$$

$$+\mathcal{L}^{-1}\left\{\frac{8(2)}{(s+3)^2+2^2}\right\}$$

$$=3e^{-3t}\cos 2t+8e^{-3t}\sin 2t$$

Hence $y=e^{-3t}(3\cos 2t+8\sin 2t)$

71.3 Worked problems on solving differential equations using Laplace transforms

Problem 4. Use Laplace transforms to solve the differential equation:

$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = e^{2x} + 20,$$ given that when $x = 0$, $y = 0$ and $\frac{dy}{dx} = -\frac{1}{3}$

Using the procedure:

(i) $$\mathcal{L}\left\{\frac{d^2y}{dx^2}\right\} - 7\mathcal{L}\left\{\frac{dy}{dx}\right\} + 10\mathcal{L}\{y\} = \mathcal{L}\{e^{2x} + 20\}$$

Hence $$[s^2\mathcal{L}\{y\} - sy(0) - y'(0)] - 7[s\mathcal{L}\{y\} - y(0)] + 10\mathcal{L}\{y\} = \frac{1}{s-2} + \frac{20}{s}$$

(ii) $y(0) = 0$ and $y'(0) = -\frac{1}{3}$

Hence $$s^2\mathcal{L}\{y\} - 0 - \left(-\frac{1}{3}\right) - 7s\mathcal{L}\{y\} + 0 + 10\mathcal{L}\{y\} = \frac{21s-40}{s(s-2)}$$

(iii) $$(s^2 - 7s + 10)\mathcal{L}\{y\} = \frac{21s-40}{s(s-2)} - \frac{1}{3}$$

$$= \frac{3(21s-40) - s(s-2)}{3s(s-2)}$$

$$= \frac{-s^2 + 65s - 120}{3s(s-2)}$$

Hence $$\mathcal{L}\{y\} = \frac{-s^2 + 65s - 120}{3s(s-2)(s^2 - 7s + 10)}$$

$$= \frac{1}{3}\left[\frac{-s^2 + 65s - 120}{s(s-2)(s-2)(s-5)}\right]$$

$$= \frac{1}{3}\left[\frac{-s^2 + 65s - 120}{s(s-5)(s-2)^2}\right]$$

(iv) $$y = \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{-s^2 + 65s - 120}{s(s-5)(s-2)^2}\right\}$$

$$\frac{-s^2 + 65s - 120}{s(s-5)(s-2)^2}$$

$$\equiv \frac{A}{s} + \frac{B}{s-5} + \frac{C}{s-2} + \frac{D}{(s-2)^2}$$

$$\equiv \frac{\begin{pmatrix} A(s-5)(s-2)^2 + B(s)(s-2)^2 \\ + C(s)(s-5)(s-2) + D(s)(s-5) \end{pmatrix}}{s(s-5)(s-2)^2}$$

71.3 Worked problems on solving differential equations using Laplace transforms

$$= \frac{-s^2+65s-120}{3s(s-2)}$$

Hence $\mathcal{L}\{y\} = \dfrac{-s^2+65s-120}{3s(s-2)(s^2-7s+10)}$

$$= \frac{1}{3}\left[\frac{-s^2+65s-120}{s(s-2)(s-2)(s-5)}\right]$$

$$= \frac{1}{3}\left[\frac{-s^2+65s-120}{s(s-5)(s-2)^2}\right]$$

(iv) $y = \dfrac{1}{3}\mathcal{L}^{-1}\left\{\dfrac{-s^2+65s-120}{s(s-5)(s-2)^2}\right\}$

$$\frac{-s^2+65s-120}{s(s-5)(s-2)^2}$$

$$\equiv \frac{A}{s}+\frac{B}{s-5}+\frac{C}{s-2}+\frac{D}{(s-2)^2}$$

$$\equiv \frac{\left(\begin{array}{c} A(s-5)(s-2)^2+B(s)(s-2)^2 \\ +C(s)(s-5)(s-2)+D(s)(s-5) \end{array}\right)}{s(s-5)(s-2)^2}$$

Hence

$-s^2+65s-120$

$$\equiv A(s-5)(s-2)^2 + B(s)(s-2)^2 + C(s)(s-5)(s-2) + D(s)(s-5)$$

When $s=0$, $-120=-20A$, from which, $A=6$

When $s=5$, $180=45B$, from which, $B=4$

When $s=2$, $6=-6D$, from which, $D=-1$

Equating s^3 terms gives: $0=A+B+C$, from which, $C=-10$

Hence $\dfrac{1}{3}\mathcal{L}^{-1}\left\{\dfrac{-s^2+65s-120}{s(s-5)(s-2)^2}\right\}$

$$= \frac{1}{3}\mathcal{L}^{-1}\left\{\frac{6}{s}+\frac{4}{s-5}-\frac{10}{s-2}-\frac{1}{(s-2)^2}\right\}$$

$$= \frac{1}{3}[6+4\,e^{5x}-10\,e^{2x}-x\,e^{2x}]$$

Thus $\boldsymbol{y = 2 + \dfrac{4}{3}e^{5x} - \dfrac{10}{3}e^{2x} - \dfrac{x}{3}e^{2x}}$

71.3 Worked problems on solving differential equations using Laplace transforms

Problem 5. The current flowing in an electrical circuit is given by the differential equation

$Ri + L(\mathrm{d}i/\mathrm{d}t) = E$, where E, L and R are constants. Use Laplace transforms to solve the equation for current i given that when $t=0$, $i=0$

Using the procedure:

(i) $\mathcal{L}\{Ri\} + \mathcal{L}\left\{L\frac{\mathrm{d}i}{\mathrm{d}t}\right\} = \mathcal{L}\{E\}$

i.e. $R\mathcal{L}\{i\} + L[s\mathcal{L}\{i\} - i(0)] = \dfrac{E}{s}$

(ii) $i(0)=0$, hence $R\mathcal{L}\{i\} + Ls\mathcal{L}\{i\} = \dfrac{E}{s}$

(iii) Rearranging gives:

$$(R+Ls)\mathcal{L}\{i\} = \frac{E}{s}$$

(ii) $i(0)=0$, hence $R\mathcal{L}\{i\} + Ls\mathcal{L}\{i\} = \dfrac{E}{s}$

(iii) Rearranging gives:

$$(R+Ls)\mathcal{L}\{i\} = \frac{E}{s}$$

i.e. $\mathcal{L}\{i\} = \dfrac{E}{s(R+Ls)}$

(iv) $i = \mathcal{L}^{-1}\left\{\dfrac{E}{s(R+Ls)}\right\}$

$$\frac{E}{s(R+Ls)} \equiv \frac{A}{s} + \frac{B}{R+Ls}$$

$$\equiv \frac{A(R+Ls)+Bs}{s(R+Ls)}$$

Hence $E = A(R+Ls)+Bs$

When $s=0$, $E = AR$,

from which, $A = \dfrac{E}{R}$

(ii) $i(0)=0$, hence $R\mathcal{L}\{i\}+Ls\mathcal{L}\{i\}=\dfrac{E}{s}$

(iii) Rearranging gives:

$$(R+Ls)\mathcal{L}\{i\}=\frac{E}{s}$$

i.e. $\mathcal{L}\{i\}=\dfrac{E}{s(R+Ls)}$

(iv) $i=\mathcal{L}^{-1}\left\{\dfrac{E}{s(R+Ls)}\right\}$

$$\frac{E}{s(R+Ls)}\equiv\frac{A}{s}+\frac{B}{R+Ls}$$

$$\equiv\frac{A(R+Ls)+Bs}{s(R+Ls)}$$

Hence $E=A(R+Ls)+Bs$

When $s=0$, $E=AR$,

from which, $A=\dfrac{E}{R}$

When $s=-\dfrac{R}{L}$, $E=B\left(-\dfrac{R}{L}\right)$

from which, $B=-\dfrac{EL}{R}$

Hence $\mathcal{L}^{-1}\left\{\dfrac{E}{s(R+Ls)}\right\}$

$$=\mathcal{L}^{-1}\left\{\frac{E/R}{s}+\frac{-EL/R}{R+Ls}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{E}{Rs}-\frac{EL}{R(R+Ls)}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{E}{R}\left(\frac{1}{s}\right)-\frac{E}{R}\left(\frac{1}{\frac{R}{L}+s}\right)\right\}$$

$$=\frac{E}{R}\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{1}{\left(s+\frac{R}{L}\right)}\right\}$$

Hence **current** $\boldsymbol{i=\dfrac{E}{R}\left(1-e^{-\frac{Rt}{L}}\right)}$

1. A first-order differential equation involving current i in a series R–L circuit is given by: $\frac{di}{dt} + 5i = \frac{E}{2}$ and $i = 0$ at time $t = 0$

   Use Laplace transforms to solve for i when (a) $E = 20$ (b) $E = 40e^{-3t}$ and (c) $E = 50 \sin 5t$

In Problems 2 to 9, use Laplace transforms to solve the given differential equations.

2. $9\frac{d^2y}{dt^2} - 24\frac{dy}{dt} + 16y = 0$, given $y(0) = 3$ and $y'(0) = 3$

3. $\frac{d^2x}{dt^2} + 100x = 0$, given $x(0) = 2$ and $x'(0) = 0$

4. $\frac{d^2i}{dt^2} + 1000\frac{di}{dt} + 250\,000i = 0$, given $i(0) = 0$ and $i'(0) = 100$

5. $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 8x = 0$, given $x(0) = 4$ and $x'(0) = 8$

6. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 3e^{4x}$, given $y(0) = -\frac{2}{3}$ and $y'(0) = 4\frac{1}{3}$

7. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 3\sin x$, given $y(0) = 0$ and $y'(0) = 0$

8. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 3\cos 3x - 11\sin 3x$, given $y(0) = 0$ and $y'(0) = 6$

9. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 3e^x\cos 2x$, given $y(0) = 2$ and $y'(0) = 5$

LAPLACE TRANSFORMS

The solution of simultaneous differential equations using Laplace transforms

Introduction

- **As stated in previous chapters, Laplace transforms have many applications in mathematics, physics, optics, electrical engineering, control engineering, signal processing, and probability and Laplace transforms and their inverses are a mathematical technique which allows us to solve differential equations, by primarily using algebraic methods.**

- **Specifically, this chapter explains the procedure for solving simultaneous differential equations; this requires all of the knowledge gained in the preceding chapters.**

At the end of this chapter, you should be able to:

- understand the procedure to solve simultaneous differential equations using Laplace transforms

- solve simultaneous differential equations using Laplace transforms

72.1 Introduction

It is sometimes necessary to solve simultaneous differential equations. An example occurs when two electrical circuits are coupled magnetically where the equations relating the two currents i_1 and i_2 are typically:

$$L_1\frac{\mathrm{d}i_1}{\mathrm{d}t}+M\frac{\mathrm{d}i_2}{\mathrm{d}t}+R_1 i_1=E_1$$

$$L_2\frac{\mathrm{d}i_2}{\mathrm{d}t}+M\frac{\mathrm{d}i_1}{\mathrm{d}t}+R_2 i_2=0$$

where L represents inductance, R resistance, M mutual inductance and E_1 the p.d. applied to one of the circuits.

72.2 Procedure to solve simultaneous differential equations using Laplace transforms

(i) Take the Laplace transform of both sides of each simultaneous equation by applying the formulae for the Laplace transforms of derivatives using a list of standard Laplace transforms,

(ii) Put in the initial conditions, i.e. $x(0)$, $y(0)$, $x'(0)$, $y'(0)$

(iii) Solve the simultaneous equations for $\mathcal{L}\{y\}$ and $\mathcal{L}\{x\}$ by the normal algebraic method.

(iv) Determine y and x by using, where necessary, partial fractions, and taking the inverse of each term.

Problem 1. Solve the following pair of simultaneous differential equations

$$\frac{dy}{dt} + x = 1$$

$$\frac{dx}{dt} - y + 4e^t = 0$$

given that at $t = 0$, $x = 0$ and $y = 0$

Using the above procedure:

(i) $$\mathcal{L}\left\{\frac{dy}{dt}\right\} + \mathcal{L}\{x\} = \mathcal{L}\{1\} \quad (1)$$

$$\mathcal{L}\left\{\frac{dx}{dt}\right\} - \mathcal{L}\{y\} + 4\mathcal{L}\{e^t\} = 0 \quad (2)$$

Equation (1) becomes:

$$[s\mathcal{L}\{y\} - y(0)] + \mathcal{L}\{x\} = \frac{1}{s} \quad (1')$$

$$[s\mathcal{L}\{x\} - x(0)] - \mathcal{L}\{y\} = -\frac{4}{s-1} \quad (2')$$

(ii) $x(0) = 0$ and $y(0) = 0$ hence

Equation (1′) becomes:

$$s\mathcal{L}\{y\} + \mathcal{L}\{x\} = \frac{1}{s} \quad (1'')$$

and equation (2′) becomes:

$$s\mathcal{L}\{x\} - \mathcal{L}\{y\} = -\frac{4}{s-1}$$

or $$-\mathcal{L}\{y\} + s\mathcal{L}\{x\} = -\frac{4}{s-1} \quad (2'')$$

(iii) $1 \times$ equation (1″) and $s \times$ equation (2″) gives:

$$s\mathcal{L}\{y\} + \mathcal{L}\{x\} = \frac{1}{s} \quad (3)$$

$$-s\mathcal{L}\{y\} + s^2\mathcal{L}\{x\} = -\frac{4s}{s-1} \quad (4)$$

Adding equations (3) and (4) gives:

$$(s^2+1)\mathcal{L}\{x\} = \frac{1}{s} - \frac{4s}{s-1}$$

$$= \frac{(s-1) - s(4s)}{s(s-1)}$$

$$= \frac{-4s^2 + s - 1}{s(s-1)}$$

$$[s\mathcal{L}\{x\} - x(0)] - \mathcal{L}\{y\} = -\frac{4}{s-1} \qquad (2')$$

(ii) $x(0) = 0$ and $y(0) = 0$ hence

Equation $(1')$ becomes:

$$s\mathcal{L}\{y\} + \mathcal{L}\{x\} = \frac{1}{s} \qquad (1'')$$

and equation $(2')$ becomes:

$$s\mathcal{L}\{x\} - \mathcal{L}\{y\} = -\frac{4}{s-1}$$

or $-\mathcal{L}\{y\} + s\mathcal{L}\{x\} = -\dfrac{4}{s-1} \qquad (2'')$

(iii) $1 \times$ equation $(1'')$ and $s \times$ equation $(2'')$ gives:

$$s\mathcal{L}\{y\} + \mathcal{L}\{x\} = \frac{1}{s} \qquad (3)$$

$$-s\mathcal{L}\{y\} + s^2\mathcal{L}\{x\} = -\frac{4s}{s-1} \qquad (4)$$

Adding equations (3) and (4) gives:

$$(s^2+1)\mathcal{L}\{x\} = \frac{1}{s} - \frac{4s}{s-1} = \frac{(s-1) - s(4s)}{s(s-1)} = \frac{-4s^2+s-1}{s(s-1)}$$

from which, $\mathcal{L}\{x\} = \dfrac{-4s^2+s-1}{s(s-1)(s^2+1)} \qquad (5)$

Using partial fractions

$$\frac{-4s^2+s-1}{s(s-1)(s^2+1)} \equiv \frac{A}{s} + \frac{B}{(s-1)} + \frac{Cs+D}{(s^2+1)} = \frac{\left(\begin{array}{c} A(s-1)(s^2+1) + Bs(s^2+1) \\ +(Cs+D)s(s-1) \end{array}\right)}{s(s-1)(s^2+1)}$$

Hence

$$-4s^2+s-1 = A(s-1)(s^2+1) + Bs(s^2+1) + (Cs+D)s(s-1)$$

When $s = 0$, $-1 = -A$ hence $\mathbf{A = 1}$

When $s = 1$, $-4 = 2B$ hence $\mathbf{B = -2}$

Equating s^3 coefficients:

$$0 = A + B + C \quad \text{hence} \quad \mathbf{C = 1}$$

(since $A = 1$ and $B = -2$)

Equating s^2 coefficients:

$$-4 = -A + D - C \quad \text{hence} \quad \mathbf{D = -2}$$

(since $A = 1$ and $C = 1$)

from which, $\mathcal{L}\{x\} = \dfrac{-4s^2 + s - 1}{s(s-1)(s^2+1)}$ (5)

Using partial fractions

$$\frac{-4s^2+s-1}{s(s-1)(s^2+1)} \equiv \frac{A}{s} + \frac{B}{(s-1)} + \frac{Cs+D}{(s^2+1)}$$

$$= \frac{\left(\begin{array}{c} A(s-1)(s^2+1) + Bs(s^2+1) \\ +(Cs+D)s(s-1) \end{array}\right)}{s(s-1)(s^2+1)}$$

Hence

$$-4s^2 + s - 1 = A(s-1)(s^2+1) + Bs(s^2+1) + (Cs+D)s(s-1)$$

When $s=0$, $-1 = -A$ hence $\mathbf{A=1}$

When $s=1$, $-4 = 2B$ hence $\mathbf{B=-2}$

Equating s^3 coefficients:

$$0 = A + B + C \quad \text{hence} \quad \mathbf{C=1}$$
(since $A=1$ and $B=-2$)

Equating s^2 coefficients:

$$-4 = -A + D - C \quad \text{hence} \quad \mathbf{D=-2}$$
(since $A=1$ and $C=1$)

Thus $\mathcal{L}\{x\} = \dfrac{-4s^2+s-1}{s(s-1)(s^2+1)}$

$$= \frac{1}{s} - \frac{2}{(s-1)} + \frac{s-2}{(s^2+1)}$$

(iv) Hence

$$x = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{(s-1)} + \frac{s-2}{(s^2+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{(s-1)} + \frac{s}{(s^2+1)} - \frac{2}{(s^2+1)}\right\}$$

i.e. $\mathbf{x = 1 - 2e^t + \cos t - 2\sin t}$,

From the second equation given in the question,

$$\frac{dx}{dt} - y + 4e^t = 0$$

from which,

$$y = \frac{dx}{dt} + 4e^t$$

$$= \frac{d}{dt}(1 - 2e^t + \cos t - 2\sin t) + 4e^t$$

$$= -2e^t - \sin t - 2\cos t + 4e^t$$

i.e. $\mathbf{y = 2e^t - \sin t - 2\cos t}$

Problem 3. Solve the following pair of simultaneous differential equations

$$\frac{d^2x}{dt^2} - x = y$$

$$\frac{d^2y}{dt^2} + y = -x$$

given that at $t=0$, $x=2$, $y=-1$, $\frac{dx}{dt}=0$ and $\frac{dy}{dt}=0$

Using the procedure:

(i) $$[s^2\mathcal{L}\{x\} - sx(0) - x'(0)] - \mathcal{L}\{x\} = \mathcal{L}\{y\} \quad (1)$$

$$[s^2\mathcal{L}\{y\} - sy(0) - y'(0)] + \mathcal{L}\{y\} = -\mathcal{L}\{x\} \quad (2)$$

(ii) $x(0)=2$, $y(0)=-1$, $x'(0)=0$ and $y'(0)=0$

hence $$s^2\mathcal{L}\{x\} - 2s - \mathcal{L}\{x\} = \mathcal{L}\{y\} \quad (1')$$

$$s^2\mathcal{L}\{y\} + s + \mathcal{L}\{y\} = -\mathcal{L}\{x\} \quad (2')$$

(iii) Rearranging gives:

$$(s^2-1)\mathcal{L}\{x\} - \mathcal{L}\{y\} = 2s \quad (3)$$

$$\mathcal{L}\{x\} + (s^2+1)\mathcal{L}\{y\} = -s \quad (4)$$

Equation $(3)\times(s^2+1)$ and equation $(4)\times 1$ gives:

$$(s^2+1)(s^2-1)\mathcal{L}\{x\} - (s^2+1)\mathcal{L}\{y\} = (s^2+1)2s \quad (5)$$

$$\mathcal{L}\{x\} + (s^2+1)\mathcal{L}\{y\} = -s \quad (6)$$

Adding equations (5) and (6) gives:

$$[(s^2+1)(s^2-1)+1]\mathcal{L}\{x\} = (s^2+1)2s - s$$

i.e. $$s^4\mathcal{L}\{x\} = 2s^3 + s = s(2s^2+1)$$

from which, $$\mathcal{L}\{x\} = \frac{s(2s^2+1)}{s^4} = \frac{2s^2+1}{s^3}$$

$$= \frac{2s^2}{s^3} + \frac{1}{s^3} = \frac{2}{s} + \frac{1}{s^3}$$

(iii) Rearranging gives:

$$(s^2-1)\mathcal{L}\{x\}-\mathcal{L}\{y\}=2s \qquad (3)$$

$$\mathcal{L}\{x\}+(s^2+1)\mathcal{L}\{y\}=-s \qquad (4)$$

Equation (3) $\times (s^2+1)$ and equation (4) $\times 1$ gives:

$$(s^2+1)(s^2-1)\mathcal{L}\{x\}-(s^2+1)\mathcal{L}\{y\} = (s^2+1)2s \qquad (5)$$

$$\mathcal{L}\{x\}+(s^2+1)\mathcal{L}\{y\}=-s \qquad (6)$$

Adding equations (5) and (6) gives:

$$[(s^2+1)(s^2-1)+1]\mathcal{L}\{x\}=(s^2+1)2s-s$$

i.e. $s^4\mathcal{L}\{x\}=2s^3+s=s(2s^2+1)$

from which, $\mathcal{L}\{x\}=\dfrac{s(2s^2+1)}{s^4}=\dfrac{2s^2+1}{s^3}$

$$=\frac{2s^2}{s^3}+\frac{1}{s^3}=\frac{2}{s}+\frac{1}{s^3}$$

(iv) Hence $x=\mathcal{L}^{-1}\left\{\dfrac{2}{s}+\dfrac{1}{s^3}\right\}$

i.e. $\qquad \mathbf{x=2+\frac{1}{2}t^2}$

Returning to equations (3) and (4) to determine y:

$1\times$ equation (3) and $(s^2-1)\times$ equation (4) gives:

$$(s^2-1)\mathcal{L}\{x\}-\mathcal{L}\{y\}=2s \qquad (7)$$

$$(s^2-1)\mathcal{L}\{x\}+(s^2-1)(s^2+1)\mathcal{L}\{y\} = -s(s^2-1) \qquad (8)$$

Equation (7) $-$ equation (8) gives:

$$[-1-(s^2-1)(s^2+1)]\mathcal{L}\{y\} = 2s+s(s^2-1)$$

i.e. $\qquad -s^4\mathcal{L}\{y\}=s^3+s$

and $\qquad \mathcal{L}\{y\}=\dfrac{s^3+s}{-s^4}=-\dfrac{1}{s}-\dfrac{1}{s^3}$

from which, $\qquad y=\mathcal{L}^{-1}\left\{-\dfrac{1}{s}-\dfrac{1}{s^3}\right\}$

i.e. $\qquad \mathbf{y=-1-\frac{1}{2}t^2}$

(iv) Hence $x = \mathcal{L}^{-1}\left\{\frac{2}{s} + \frac{1}{s^3}\right\}$

i.e. $\mathbf{x = 2 + \frac{1}{2}t^2}$

Returning to equations (3) and (4) to determine y:

$1 \times$ equation (3) and $(s^2 - 1) \times$ equation (4) gives:

$$(s^2 - 1)\mathcal{L}\{x\} - \mathcal{L}\{y\} = 2s \qquad (7)$$

$$(s^2 - 1)\mathcal{L}\{x\} + (s^2 - 1)(s^2 + 1)\mathcal{L}\{y\} = -s(s^2 - 1) \qquad (8)$$

Equation (7) $-$ equation (8) gives:

$$[-1 - (s^2 - 1)(s^2 + 1)]\mathcal{L}\{y\} = 2s + s(s^2 - 1)$$

i.e. $-s^4\mathcal{L}\{y\} = s^3 + s$

and $\mathcal{L}\{y\} = \frac{s^3 + s}{-s^4} = -\frac{1}{s} - \frac{1}{s^3}$

from which, $y = \mathcal{L}^{-1}\left\{-\frac{1}{s} - \frac{1}{s^3}\right\}$

i.e. $\mathbf{y = -1 - \frac{1}{2}t^2}$

When $s = 0$, $-6 = -2A$, hence $\mathbf{A = 3}$

When $s = 1$, $0 = 3C$, hence $\mathbf{C = 0}$

When $s = -2$, $30 = 6B$, hence $\mathbf{B = 5}$

Thus $\mathcal{L}\{x\} = \frac{8s^2 - 2s - 6}{s(s+2)(s-1)} = \frac{3}{s} + \frac{5}{(s+2)}$

Hence $x = \mathcal{L}^{-1}\left\{\frac{3}{s} + \frac{5}{s+2}\right\} = \mathbf{3 + 5e^{-2t}}$

Therefore the solutions of the given simultaneous differential equations are

$$\mathbf{y = 1 + 2e^{-2t}} \quad \text{and} \quad \mathbf{x = 3 + 5e^{-2t}}$$

Solve the following pairs of simultaneous differential equations:

1. $2\frac{dx}{dt}+\frac{dy}{dt}=5\,e^{t}$

$\frac{dy}{dt}-3\frac{dx}{dt}=5$

given that when $t=0$, $x=0$ and $y=0$

2. $2\frac{dy}{dt}-y+x+\frac{dx}{dt}-5\sin t=0$

$3\frac{dy}{dt}+x-y+2\frac{dx}{dt}-e^{t}=0$

given that at $t=0$, $x=0$ and $y=0$

3. $\frac{d^2x}{dt^2}+2x=y$

$\frac{d^2y}{dt^2}+2y=x$

given that at $t=0$, $x=4$, $y=2$, $\frac{dx}{dt}=0$

and $\frac{dy}{dt}=0$